R-charges, Chiral rings and RG flows in supersymmetric Chern-Simons-Matter theories

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ABSTRACT: We discuss the non-perturbative behavior of the $U(1)_R$ symmetry in $\mathcal{N} = 2$ superconformal Chern-Simons theories coupled to matter in the (anti)fundamental and adjoint representations of the gauge group, which we take to be $U(N)$. Inequalities constraining this behavior are obtained as consequences of spontaneous breaking of supersymmetry and Seiberg duality. This information reveals a web of RG flows connecting different interacting superconformal field theories in three dimensions. We observe that a subclass of these theories admits an ADE classification. In addition, we postulate new examples of Seiberg duality in $\mathcal{N} = 2$ and $\mathcal{N} = 3$ Chern-Simons-matter theories and point out interesting parallels with familiar non-perturbative properties in $\mathcal{N} = 1$ (adjoint) SQCD theories in four dimensions where the exact $U(1)_R$ symmetry can be determined using a-maximization.

Contents

1 Introduction

Three-dimensional gauge theories exhibit a range of interesting, non-perturbative phenomena (see, for example, [1–6] and references/citations thereof). Some of the complicating features of these theories stem from the fact that the gauge interaction is a classically relevant operator in three dimensions. As we flow towards the infrared (IR), the gauge coupling grows indefinitely and the theory becomes strongly coupled. A way to ameliorate this strong coupling problem is to add to the Lagrangian the Chern-Simons (CS) interaction

$$\mathcal{S}_{\mathrm{CS}} = \frac{k}{4\pi} \int \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \tag{1.1}$$

where A is the gauge field one-form and the constant k is the CS level. This interaction, which is specific to three dimensions, makes the gauge field massive with a mass of order

$$m_{\mathrm{CS}} \sim g_{\mathrm{YM}}^2 k \tag{1.2}$$

where g_{YM} is the gauge coupling. At energies below the scale set by m_{CS} the IR behavior of the theory is controlled by the CS interaction and the flow towards strong coupling is effectively cutoff.

Chern-Simons theories coupled to matter are non-trivial quantum field theories. Without matter the Chern-Simons Lagrangian (1.1) defines a topological quantum field theory with a fascinating connection to two-dimensional Wess-Zumino-Witten (WZW) models [7].

In this paper we will discuss Chern-Simons-Matter (CSM) theories with at least four supercharges, that is $\mathcal{N} = 2$ supersymmetry in three dimensions. These theories are characterized by a gauge group G, the CS level k and the representations R_i of the matter fields. The gauge field is part of the $\mathcal{N} = 2$ vector multiplet V and matter is organized in $\mathcal{N} = 2$ chiral multiplets Φ_i. The components of $\mathcal{N} = 2$ multiplets can be deduced easily by dimensional reduction from $\mathcal{N} = 1$ multiplets in four dimensions. The vector multiplet contains the three-dimensional gauge field A_μ, a scalar σ, an auxiliary scalar D and a two-component Dirac spinor χ. A chiral multiplet Φ_i contains a complex scalar φ_i and a Dirac spinor ψ_i.

The $\mathcal{N} = 2$ supersymmetric CS action has a known expression in superspace language [8–11]. This expression becomes simpler in Wess-Zumino (WZ) gauge

$$\mathcal{S}_{\mathrm{CS}}^{\mathcal{N}=2} = \frac{k}{4\pi} \int d^3x \, \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A - \bar{\chi}\chi + 2D\sigma \right) . \tag{1.3}$$

The matter multiplets have the standard kinetic terms

$$\mathcal{S}_{\mathrm{matter,kin}} = \int d^3x \, d^4\theta \, \sum_i \bar{\Phi}_i e^V \Phi_i . \tag{1.4}$$

Further coupling via superpotential interactions is possible.

Integrating out the auxiliary fields and massive fermions of the $\mathcal{N} = 2$ vector multiplet gives an interacting theory of scalars and fermions [12]. Besides their intrinsic interest, such theories have important applications in M-theory and the AdS/CFT correspondence.

Conformal CSM theories with enhanced supersymmetry ($\mathcal{N} = 4, 5, 6, 8$) have been argued [13–17] to provide a gauge theory description of the infrared dynamics of M2-branes in different setups and to be related by holography to string/M-theory on AdS$_4$ backgrounds. In this paper, our main objective are the field theory properties of $\mathcal{N} = 2$ CSM theories, although some comments will be made on aspects related to M-brane dynamics and the AdS$_4$/CFT$_3$ correspondence.

Assuming the absence of superpotential interactions, all couplings in the action

$$\mathcal{S}_{\text{CSM}}^{\mathcal{N}=2} = \mathcal{S}_{\text{CS}}^{\mathcal{N}=2} + \mathcal{S}_{\text{matter,kin}} \tag{1.5}$$

are controlled by the inverse of the CS level $\frac{1}{k}$. The theory is weakly coupled at large k. For non-abelian gauge groups G the level k is an integer, and does not receive quantum corrections, except for a possible one-loop shift [11, 18, 19]. In fact, one can argue that these theories are both classically and quantum mechanically superconformal [12]. In these superconformal theories the U$(1)_R$ symmetry is non-trivial and depends on k. In other words, the scaling dimensions of chiral operators receive k-dependent anomalous dimensions.

The addition of relevant superpotential interactions to the action (1.5) breaks the conformal invariance and generates a renormalization group (RG) flow towards new interacting fixed points. Identifying these flows and determining analytically their properties is, in general, a non-perturbative question that remains largely open.

In order to make progress in this problem it is desirable to determine with exact analytical methods the U$(1)_R$ symmetry in any of the above fixed points. Knowing this symmetry would allow us to determine the exact scaling dimension of chiral operators. Relevant chiral operators can be added to the Lagrangian as superpotential deformations to generate new RG flows and IR fixed points.

A similar question can be posed in four-dimensional gauge theories with $\mathcal{N} = 1$ supersymmetry. In this context, the exact U$(1)_R$ symmetry can be determined with a combination of a-maximization and 't Hooft anomaly matching [20] .

There are several tools that allow us to compute non-perturbative quantities in four-dimensional $\mathcal{N} = 1$ gauge theories. Many of them are closely related to the presence of anomalies. The NSVZ exact β-function formula [21], a-maximization [20] and 't Hooft anomaly matching are well known examples. In some cases, additional information can be obtained with the use of Seiberg duality [22], which is a powerful strong/weak coupling duality. Since there are no anomalies of continuous symmetries in three dimensions a corresponding understanding of the properties of three-dimensional quantum field theories is currently lacking.

In an effort to obtain a more precise understanding of the properties of U$(1)_R$ symmetries and RG flows in $\mathcal{N} = 2$ CSM theories, we will examine in this paper what happens in U(N_c) $\mathcal{N} = 2$ CS theories coupled to a set of matter fields that contains: N_f chiral superfields Q^i ($i = 1, \ldots, N_f$) in the fundamental representation, N_f chiral superfields $\widetilde{Q}_i$ in the antifundamental and zero, one or two chiral superfields in the adjoint representation of the gauge group. Non-trivial RG flows can be generated in these theories with superpotential deformations that involve gauge invariant chiral operators made out of the above fields.

In the presence of superpotential interactions these theories can be viewed as three-dimensional versions of $\mathcal{N} = (2,2)$ Landau-Ginzburg (LG) models in two dimensions. They can also be viewed as three-dimensional versions of $\mathcal{N} = 1$ SQCD theories in four dimensions with N_f flavor chiral multiplets and zero, one or two adjoints.[1] Because of the many similarities with the four-dimensional (adjoint) SQCD theories we will sometimes call the corresponding CSM theories CS-SQCD, 1-adjoint CS-SQCD and 2-adjoint CS-SQCD.

We will find that the similarities between CS-SQCD and SQCD theories in three and four dimensions respectively are not limited to the matter content but extend to non-perturbative aspects of their dynamics. Some of the common features can be traced back to the similarities between the chiral rings. Other features, however, like the stability of the supersymmetric vacuum and Seiberg duality, are highly non-trivial and appear to arise in three and four dimensions through different mechanisms. These similarities are impressive and reveal how rich the dynamics of $\mathcal{N} = 2$ CSM theories is.

Spontaneous supersymmetry breaking and Seiberg duality are properties that arise in some of the theories we will examine. They can be inferred most easily from the rules of brane dynamics in configurations of D-branes and NS5-branes in type IIB string theory [23–26]. In some cases, where a string theory construction is unknown, we will argue for bounds of spontaneous breaking of supersymmetry using the chiral ring structure directly in field theory. These properties have important consequences for the exact $U(1)_R$ symmetry in these theories and as such they will provide a useful semi-quantitative guide to the behavior of R-charges as we move from the weak to the strong coupling regime. This indirect reasoning is what will allow us to make some progress despite our lack of an exact analytic tool in three dimensions like a-maximization.

With this information about R-symmetries we will be able to identify a web of RG flows connecting different conformal field theories (CFTs) in three dimensions. In the IR of these flows interacting fixed points of the β-function arise from a balancing of two counteracting sources: the gauge interactions that work to decrease the R-charges and the superpotential interactions that work to increase them. This can be verified explicitly in some cases with a two-loop computation in perturbation theory. Furthermore, we find that a subset of the CFTs arising in this way admits an ADE classification. A similar classification was also observed in four-dimensional two-adjoint SQCD theories [27].

The organization of this paper is as follows. Section 2 reviews the key results obtained in ref. [25] about CS-SQCD theories and supplements them with new observations on the phase structure and the $U(1)_R$ symmetry. Section 3 considers RG flows that arise in CS-SQCD theories with superpotential deformations that involve quadratic and cubic meson interactions. Section 4 is devoted to the 1-adjoint CS-SQCD theories reviewing and extending the results obtained in ref. [26]. Special emphasis is given to the R-charge of the adjoint chiral superfield as a function of the parameters of the theory. Sections 5 and 6 discuss RG flows that arise in 2-adjoint CS-SQCD theories with superpotential

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[1]In four dimensions two is the maximum number of adjoints if we require asymptotic freedom. We do not have a corresponding restriction in three dimensions, but since we want to compare the properties of three and four-dimensional gauge theories we will also restrict our discussion to CSM theories with up to two adjoint chiral superfields.

deformations that involve single trace, and in some cases also mesonic, chiral operators. In the process, we encounter a new set of Chern-Simons theories that exhibit Seiberg duality and the emergence of a partial ADE classification of fixed points. We conclude with an overall discussion and a list of interesting open problems in section 7. Some useful technical details are relegated to two appendices at the end of the paper.

2 CS-SQCD

2.1 Definition and phase structure

The first set of theories in our agenda are Chern-Simons versions of the four-dimensional $\mathcal{N} = 1$ SQCD theories. The Lagrangian that defines these theories consists of the $\mathcal{N} = 2$ CS interaction at level k and the $\mathcal{N} = 2$ kinetic terms for N_f pairs of $\mathcal{N} = 2$ chiral multiplets Q^i, $\widetilde{Q}_i$ ($i = 1, 2, \ldots, N_f$) in the fundamental (resp. antifundamental) representation of the gauge group. Explicit expressions for this Lagrangian can be found, for example, in ref. [12]. We will consider the unitary gauge group $G = \mathrm{U}(N_c)$. In contrast to what happens in four dimensions, here the $\mathrm{U}(1) \subset \mathrm{U}(N_c)$ is interacting and will have some important implications that will be discussed below.

We will restrict the level k to be positive. Negative values of k can be obtained by a parity transformation $x^\mu \to -x^\mu$ that effectively sends $k \to -k$. One should think of $1/k$ as the 'gauge coupling' of the CS theory. In the large N_c limit, the ratio $\lambda = \frac{N_c}{k}$ plays the rôle of the 't Hooft coupling. For simplicity, in what follows, we will consider our theories in the large-N limit where $k, N_c, N_f \gg 1$ with the ratios λ and $x = \frac{N_c}{N_f}$ kept finite. This is not necessary for most of the statements that we make below. When finite-N effects do not modify the qualitative picture it will be more convenient to think in terms of continuous, instead of discrete, parameters.

The so-defined CS-SQCD theories are superconformal both classically and quantum mechanically [12]. The global symmetry of the theory is $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f) \times \mathrm{U}(1)_a \times \mathrm{U}(1)_R$.

As we increase the number of colors N_c, say for fixed k and N_f, we encounter a critical point where supersymmetry gets spontaneously broken. At the critical point

$$N_c = N_f + k \ . \tag{2.1}$$

The simplest way to obtain this spontaneous breaking of supersymmetry is by embedding the theory in a configuration of D-branes and NS5-branes in type IIB string theory. The relevant setup was analyzed in ref. [25] and we briefly review it here for completeness. It consists of (see also the configuration (a) in figure 1)

$$
\begin{array}{llllllllll}
1 \ NS5 & : & 0 \ 1 \ 2 \ 3 \ 4 \ 5 \\
1 \ (1,k) & : & 0 \ 1 \ 2 \ \begin{bmatrix} 3 \\ 7 \end{bmatrix}_\theta \ 8 \ 9 \\
N_c \ D3 & : & 0 \ 1 \ 2 \ |6| \\
N_f \ D5 & : & 0 \ 1 \ 2 \ 7 \ 8 \ 9
\end{array}
\tag{2.2}
$$
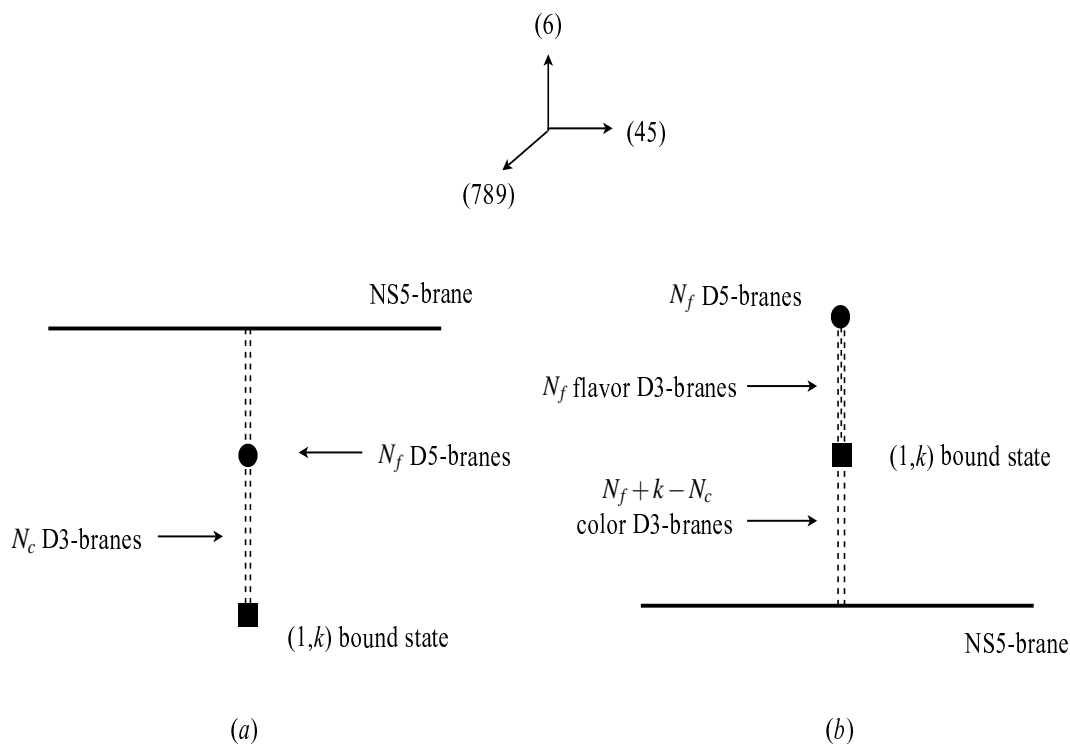
Figure 1. Configuration (a) engineers the electric version of the $\mathcal{N} = 2$ CS-SQCD theory. Configuration (b) engineers the magnetic version.

$(1, k)$ denotes the bound state of one NS5-brane and k D5-branes. $\left[\begin{smallmatrix} 3 \\ 7 \end{smallmatrix}\right]_\theta$ denotes an orientation in the (37) plane at an angle θ with respect to the axis 3. In the above configuration θ is an angle fixed by k by the relation [28]

$$\tan \theta = g_s k \ . \tag{2.3}$$

$|6|$ denotes that the D3-branes have a finite length along the x^6 direction. The CS-SQCD theory arises in this setup as the effective low-energy description of the dynamics of the theory that resides in the three-dimensional intersection of this configuration [29].

In this context, the condition for the existence of a supersymmetric vacuum is a consequence of the s-rule of brane dynamics [23, 24]. In figure 1(a) the s-rule constrains the number of D3-branes stretching directly between the NS5-brane and the $(1, k)$ bound state. This number $(N_c - N_f)$ must be smaller or equal to k in order to preserve supersymmetry, hence the condition

$$N_c \leq N_f + k \ . \tag{2.4}$$

In terms of λ and x this condition reads

$$\lambda \leq \frac{x}{x - 1} \ . \tag{2.5}$$

Accordingly, the phases of CS-SQCD are plotted in figure 2. There are two basic regions: a region where the vacuum is supersymmetric and a region where supersymmetry

Figure 2. Phases of CS-SQCD in a (λ, x) diagram. The blue region at the bottom is the perturbative region of the electric theory. The orange region at the right corner is where the topological $\mathcal{N} = 2$ CS theory is recovered.

is spontaneously broken. The supersymmetric region has a corner at $x \to \infty$, $\lambda \in [0,1)$ and a corner at $x \in [0,1)$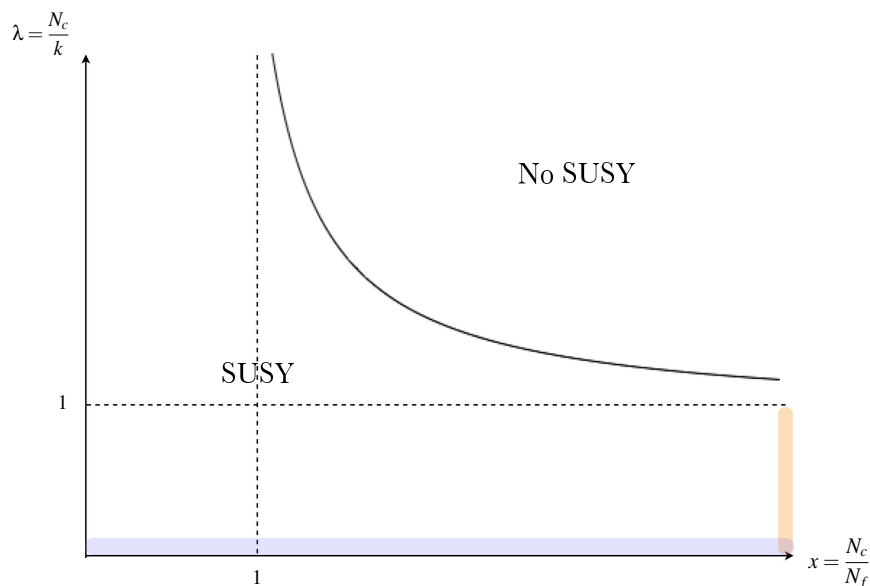, $\lambda \to \infty$. The first corner, colored orange in figure 2, is the place where $N_f \to 0$ and the theory becomes topological. At the second corner the coupling λ is infinitely large.

It is interesting to note that the value $x = 1$ is also special for the $\mathrm{SU}(N_c)$ SQCD theory in four dimensions. The IR behavior of SQCD is characterized by the single parameter x and the supersymmetric vacuum is lifted for $N_f < N_c$, *i.e.* for $x > 1$. In CS-SQCD we have instead two parameters characterizing the theory, λ and x. From the point of view of the stability of the supersymmetric vacuum, SQCD is similar to the infinitely strongly coupled regime of CS-SQCD where $\lambda \gg 1$. It is unclear whether this observation implies a deeper connection between three- and four-dimensional dynamics.

Finally, there are two regions where CS-SQCD admits a weakly coupled description. An obvious one is the region at the bottom, which is colored blue in figure 2, where $\lambda \ll 1$. The other one is the supersymmetric region close to the critical SUSY breaking curve, *i.e.* the region with $\lambda \lesssim \frac{x}{x-1}$, $x \geq 1$. Here, a weakly coupled description exists in terms of a Seiberg dual magnetic theory.

2.2 Seiberg duality

It has been proposed that the $\mathcal{N} = 2$ CS-SQCD theories exhibit Seiberg duality [25]. The dual magnetic theory is an $\mathcal{N} = 2$ CSM theory with gauge group $\mathrm{U}(N_f + k - N_c)$ at level k. It has N_f pairs of quarks $q_i, \tilde{q}^i$ ($i = 1, 2, \ldots, N_f$) and a set of magnetic meson chiral superfields M_j^i which are gauge singlets. As in four-dimensional SQCD, the magnetic

theory possesses a cubic superpotential

$$W_{\text{mag}} = M^i_j q_i \tilde{q}^j \ . \tag{2.6}$$

In three dimensions, this is a classically relevant interaction. It generates an RG flow and in the IR, where the magnetic theory is dual to the electric, this interaction becomes marginal.

To obtain Seiberg duality in the context of the string theory configuration of figure 1 we displace sequentially the N_f D5-branes and the $(1, k)$ bound state along the x^6 direction past the NS5-brane. When N_f D5-branes pass through the NS5-brane, N_f D3-branes are created. Similarly, when the $(1, k)$ bound state passes through the NS5-brane, k D3-branes are created. At the end of the process we obtain the configuration in figure 1(b) whose low-energy dynamics is described by the magnetic theory presented above.

As a small parenthesis we note that in the topological limit $N_f \to 0$, Seiberg duality has a natural interpretation as level-rank duality in the bosonic $\text{SU}(N)_k$ WZW model [17]. Indeed, at large k we can integrate out the gluino field (whose mass is proportional to k) to obtain pure CS theory at the shifted level [30]

$$k' = k - N_c \ . \tag{2.7}$$

By the CS-WZW correspondence of [7] this theory is equivalent to the (chiral) $\text{SU}(N_c)_{k-N_c}$ WZW model (for the moment we set the U(1) part of the gauge group aside). Level-rank duality implies that this is equivalent to the $\text{SU}(k - N_c)_{N_c}$ WZW model, which, again by the CS-WZW correspondence, is equivalent to the $\text{SU}(k - N_c)$ pure CS theory at level N_c. Integrating in the gluinos (and putting back the U(1) part) we recover the Seiberg dual $\mathcal{N} = 2$ CS theory with gauge group $\text{U}(k - N_c)$ and level k.

For $x > 1$ Seiberg duality acts as a strong/weak coupling duality. Indeed, the magnetic 't Hooft coupling is

$$\widetilde{\lambda} = \frac{N_f + k - N_c}{k} = 1 - \lambda \left(1 - \frac{1}{x} \right) \ . \tag{2.8}$$

Hence, when $\lambda \ll 1$, $\widetilde{\lambda} \sim 1$ and the magnetic description is strongly coupled. Conversely, when $\widetilde{\lambda} \ll 1$, $\lambda \sim \left(1 - x^{-1} \right)^{-1} > 1$ and the electric theory is strongly coupled. This strong/weak relation disappears for $x < 1$. In this case, both descriptions become strongly coupled simultaneously.

2.3 Qualitative features of R-charges

We now come to one of the central questions in this paper — how the R-charges behave as we change the parameters of the CSM theory. In the electric version of CS-SQCD the flavor symmetry $\text{SU}(N_f) \times \text{SU}(N_f)$ guarantees that all the flavors Q^i, $\widetilde{Q}_i$ ($i = 1, 2, \ldots, N_f$) have the same $\text{U}(1)_R$ charge $R_Q = R_Q(N_c, N_f, k)$. In the large-N limit $R_Q = R_Q(\lambda, x)$. Similarly, in the magnetic theory all the flavors q_i, $\tilde{q}^i$ have the same R-charge $R_q = R_q(\lambda, x)$. The magnetic theory also possesses the gauge singlet elementary meson superfield M^i_j, whose R-charge we denote as $R_M = R_M(\lambda, x)$.

The three functions R_Q, R_q and R_M are related by Seiberg duality. The composite meson chiral superfields of the electric theory $\mathcal{M}^i_j = Q^i \widetilde{Q}_j$ are mapped to the elementary

fields M_j^i. Hence,

$$R_M = 2R_Q \ . \tag{2.9}$$

Moreover, in the IR of the magnetic theory the superpotential W_{mag} (2.6) becomes marginal and

$$R_M + 2R_q = 2 \ . \tag{2.10}$$

We conclude that there is one independent R-charge function, say R_Q, and

$$R_q = 1 - R_Q \, , \quad R_M = 2R_Q \ . \tag{2.11}$$

In the four-dimensional $SU(N_c)$ SQCD we can determine R_Q exactly by demanding anomaly cancellation. The result is $R_Q = 1-x$. In three dimensions there are no anomalies for continuous symmetries, hence we cannot proceed in the same way.

Alternatively, if we did not know about anomalies in four dimensions, but we knew about Seiberg duality, it would still have been possible to determine R_Q exactly in $\mathcal{N} = 1$ SQCD. By matching the baryon operators (a_i are gauge indices here)

$$B^{i_1 \cdots i_{N_c}} = \epsilon^{a_1 a_2 \cdots a_{N_c}} Q_{a_1}^{i_1} \cdots Q_{a_{N_c}}^{i_{N_c}} \, , \qquad \widetilde{B}^{i_1 \cdots i_{N_c}} = \epsilon_{a_1 a_2 \cdots a_{N_c}} \widetilde{Q}_{i_1}^{a_1} \cdots \widetilde{Q}_{i_{N_c}}^{a_{N_c}} \tag{2.12}$$

to their magnetic duals one finds independently a new relation between R_Q and R_q

$$N_c R_Q = (N_f - N_c) R_q \tag{2.13}$$

which determines R_Q as above in agreement with the anomaly cancellation condition. We cannot repeat this exercise in CS-SQCD, because the gauge group is $U(N_c)$ and there are no baryons to match. Hence, the $U(1)$ part of the gauge group, which is so crucial for the validity of Seiberg duality in CS-SQCD,[2] is also the reason why the exact form of the function R_Q avoids a simple detection.

Some information about the behavior of $R_Q(\lambda, x)$ can be obtained in perturbative regimes. For $\lambda, \frac{1}{x} \ll 1$ a two-loop perturbative computation of R_Q gives [12] (we have taken the large-N limit for simplicity)

$$R_Q(\lambda, x) = \frac{1}{2} - \frac{\lambda^2}{16} + \cdots \tag{2.14}$$

where the dots $\cdots$ indicate subleading corrections in λ and $\frac{1}{x}$. As usual, we observe that gauge interactions work to reduce the classical R-charge.

At strong coupling the behavior of the theory depends, as we said, on the value of x. For $x < 1$, the coupling λ can grow to infinity and there is no obvious regime which admits a weakly coupled description. For $x > 1$, however, Seiberg duality provides a weakly coupled description when $\lambda \lesssim \frac{x}{x-1}$. In this regime we can compute using the magnetic theory. At the critical coupling $\lambda = \frac{x}{x-1}$, the rank of the dual gauge group vanishes and the quark superfields disappear. The magnetic superpotential is absent and the superfields M_j^i are free fields. This picture is supported by the brane setup in figure 1(b). At $\lambda = \frac{x}{x-1}$ the color D3-branes are absent, there are no quark superfields from color-flavor open strings

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[2]Indeed, if we simply dropped the $U(1)$ in CS-SQCD, baryon matching would have been problematic.
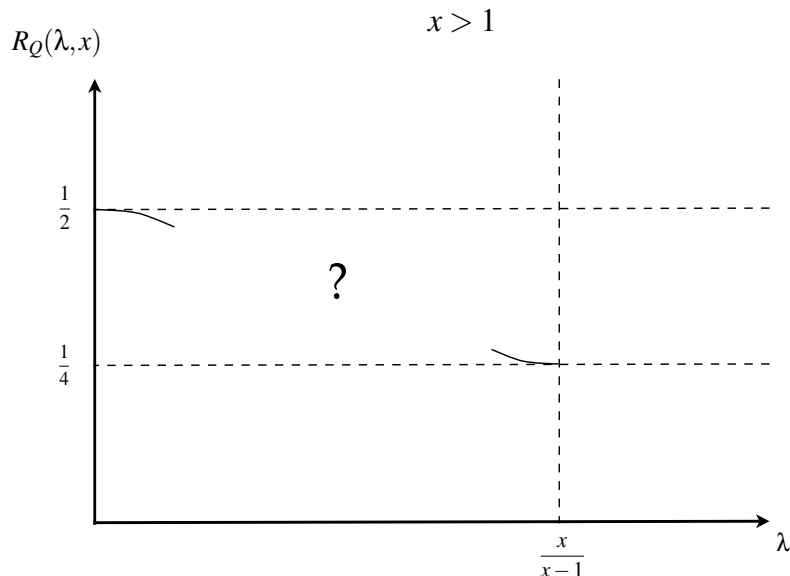
Figure 3. A plot of R_Q as a function of λ for fixed $x > 1$. R_Q interpolates between its classical value $\frac{1}{2}$ and the value $\frac{1}{4}$ at the critical point $\frac{x}{x-1}$ beyond which the theory exhibits spontaneous breaking of supersymmetry. The exact behavior of the function in this interval is currently unknown.

and the IR dynamics is dominated by the free fields M_j^i which arise from flavor-flavor open strings. Consequently, in this limit $R_M = \frac{1}{2}$ and $R_Q = \frac{1}{4}$. Moreover, at any value of $\lambda < \frac{x}{x-1}$ we must have $R_M \geq \frac{1}{2}$ by unitarity.

Assuming R_Q is a continuous function of λ the emerging picture for $x > 1$ is depicted in figure 3. At the lower part of the interval $\left[0, \frac{x}{x-1}\right]$ R_Q starts off at its classical value $\frac{1}{2}$ and then decreases. At the upper end, R_Q tends from above to the value $\frac{1}{4}$ where the mesons saturate the unitarity bound.[3] In this range R_Q is necessarily greater than $\frac{1}{4}$, but how it behaves more precisely is currently unclear. It would be interesting, for example, to know if $R_Q(\lambda, x)$ is a monotonic function of λ for fixed x (this seems to be a natural expectation in the absence of superpotential interactions). It would also be interesting to know what happens when $x < 1$.

3 CS-SQCD theories with mesonic superpotentials

3.1 Relevant mesonic superpotentials

We can deform the superconformal $\mathcal{N} = 2$ CS-SQCD theories by adding to the Lagrangian superpotential interactions that involve the chiral meson operators. For $x > 1$ the quali-

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[3]The continuity of R_Q at the critical coupling $\lambda = \frac{x}{x-1}$ is not immediately obvious from the magnetic theory point of view. Although a discontinuity is unlikely there in our opinion, a clear justification of this point would be desirable. A discontinuity of R_Q at the critical coupling would imply that R_Q tends to a finite value between $\frac{1}{2}$ and $\frac{1}{4}$ as $\lambda \to \frac{x}{x-1}$, but jumps discontinuously to the value $\frac{1}{4}$ at $\lambda = \frac{x}{x-1}$. We proceed in the next section assuming this discontinuous behavior does not occur.

tative picture of the previous section helps us understand which are the relevant superpotential interactions that we can add.

The superpotential deformations

$$\delta W_1 = m_i^j Q^i \widetilde{Q}_j \,, \qquad\qquad \delta W_2 = \frac{\alpha_2}{2}(Q^i \widetilde{Q}_j)(Q^j \widetilde{Q}_i) \qquad\qquad (3.1)$$

are relevant already at weak coupling λ. The first is a mass deformation, the second is a deformation that drives the theory to an $\mathcal{N} = 3$ supersymmetry enhanced IR fixed point. This RG flow will be discussed in the next subsection.

Higher powers of the meson operators are classically irrelevant operators. We have seen, however, that as we increase the coupling the R-charge R_Q goes down achieving the minimum value $\frac{1}{4}$ at $\lambda = \frac{x}{x-1}$. Consequently, there is a critical value of λ beyond which the sextic superpotential deformation

$$\delta W_3 = \alpha_3 (Q\widetilde{Q})^3 \qquad\qquad (3.2)$$

becomes relevant. We will discuss this deformation in subsection 3.3.

The next (and last in the supersymmetric interval) power of mesons $(Q\widetilde{Q})^4$ becomes marginal at the critical curve $\lambda = \frac{x}{x-1}$. If Seiberg duality is to be trusted there, this point has a dual description in terms of a WZ model for M with superpotential

$$W_{\text{mag}} = \alpha_4 M^4 \qquad\qquad (3.3)$$

The leading two-loop correction to the β-function of α_4 is positive (as follows quite generally from unitarity). Hence, at least perturbatively near $\alpha_4 = 0$, this perturbation is irrelevant and does not lead to a new fixed point.

3.2 Quartic deformations

The quartic deformation δW_2 in (3.1) is classically marginal, but quantum mechanically relevant. At small λ, the large-N limit β-function for the coupling α_2 is [12]

$$\frac{d\alpha_2}{dt} = \frac{N_c^2}{(8\pi)^2} \alpha_2 \left[\alpha_2^2 - \left(\frac{4\pi}{k} \right)^2 \right] \,, \qquad\qquad (3.4)$$

where t is the logarithm of the RG scale. Perturbing by δW_2 drives the theory to a new IR fixed point with α_2 value

$$\alpha_{2,\mathcal{N}=3} = \pm \frac{4\pi}{k} \qquad\qquad (3.5)$$

controlled by the CS level k. This is a fixed point with enhanced $\mathcal{N} = 3$ supersymmetry [12] (see also below). Accordingly, the new R-symmetry is SU(2). The $\text{SU}(N_f) \times \text{SU}(N_f)$ global flavor symmetry is broken down to the diagonal $\text{SU}(N_f)$ by the superpotential interaction.

At the $\mathcal{N} = 3$ fixed point the superpotential interaction is again marginal and R_Q has recovered its classical value $\frac{1}{2}$. The absence of quantum corrections to R_Q as we change λ, x at this point is consistent with the fact that the $\text{U}(1)_R$ symmetry is now part of the larger $\text{SU}(2)_R$ symmetry which precludes the presence of anomalous dimensions for the meson operators $Q\widetilde{Q}$.

As a deformation of the $\mathcal{N} = 2$ CS-SQCD theory, the $\mathcal{N} = 3$ fixed points are still expected to exhibit Seiberg duality in terms of a $U(N_f + k - N_c)$ theory and spontaneous supersymmetry breaking at $N_c = N_f + k$. In what follows we will describe two different brane configurations in type IIB string theory that confirm these properties.

3.2.1 Quartic couplings from brane configurations I

The first configuration of branes in type IIB string theory that we want to consider is similar to the configuration appearing in figure 1(a). The only difference is a change in the orientation of the $(1, k)$ bound state in the (48) and (59) planes. D3, D5, NS5 and $(1, k)$ branes are now oriented in the following way

$$
\begin{aligned}
1 \; NS5 \; &: \quad 0 \; 1 \; 2 \; 3 \; 4 \; 5 \\
1 \; (1, k) \; &: \quad 0 \; 1 \; 2 \; \begin{bmatrix} 3 \\ 7 \end{bmatrix}_\theta \; \begin{bmatrix} 4 \\ 8 \end{bmatrix}_\theta \; \begin{bmatrix} 5 \\ 9 \end{bmatrix}_{-\theta} \\
N_c \; D3 \; &: \quad 0 \; 1 \; 2 \; |6| \\
N_f \; D5 \; &: \quad 0 \; 1 \; 2 \; 7 \; 8 \; 9
\end{aligned}
\tag{3.6}
$$

θ is still the k-controlled angle that appears in eq. (2.3). This configuration expressly preserves $\mathcal{N} = 3$ supersymmetry in three dimensions [29].

The low-energy description of this setup is in terms of a $U(N_c)$ $\mathcal{N} = 3$ CS theory at level k coupled to N_f pairs of (anti)fundamentals Q^i, $\widetilde{Q}_i$ and a massive chiral superfield X in the adjoint representation of the gauge group. The Lagrangian includes the superpotential interactions

$$
W_{\mathcal{N}=3} = -\frac{k}{8\pi} \operatorname{Tr} X^2 + \widetilde{Q}_i X Q^i \; .
\tag{3.7}
$$

At large k the superfield X can be integrated out to recover the quartic superpotential δW_2 with $\alpha_2 = \frac{4\pi}{k}$.

Within string theory Seiberg duality follows, as in section 2, by moving the D5-branes and the $(1, k)$ bound state through the NS5-brane along x^6 to obtain a configuration of the form depicted in figure 1(b). The dual gauge theory is a $U(N_f + k - N_c)$ $\mathcal{N} = 3$ CS theory at level k with N_f pairs of (anti)fundamentals q_i, $\tilde{q}^i$, a massive adjoint chiral superfield X and the superpotential (3.7) with $Q, \widetilde{Q}$ replaced by $q, \tilde{q}$. From the s-rule of brane dynamics we deduce that there is no supersymmetric vacuum for $N_c > N_f + k$.

Seiberg duality in this case acts in a self-similar way. A generalization to theories with superpotentials of the form (3.7), but arbitrary power for X ($\operatorname{Tr} X^{n+1}$), will be considered in section 6.2.

3.2.2 Quartic couplings from brane configurations II

The generic deformation by δW_2 can be obtained in string theory within the following type IIB configuration

$$
\begin{aligned}
1 \ NS5 &: \quad 0\ 1\ 2\ 3\ 4\ 5 \\
1 \ (1,k) &: \quad 0\ 1\ 2\ \begin{bmatrix} 3 \\ 7 \end{bmatrix}_\theta \ 8\ 9 \\
N_c \ D3 &: \quad 0\ 1\ 2\ |6| \\
N_f \ D5 &: \quad 0\ 1\ 2\ 7\ \begin{bmatrix} 4 \\ 8 \end{bmatrix}_\psi \ \begin{bmatrix} 5 \\ 9 \end{bmatrix}_\psi
\end{aligned}
\tag{3.8}
$$

The rotation of the D5-branes by an arbitrary angle ψ in the (48), (59) planes provides the quartic coupling δW_2. Similar type IIA configurations related to four-dimensional $\mathcal{N} = 1$ SQCD theories with quartic superpotential have been discussed in [31] (an earlier discussion oriented also towards three-dimensional $\mathcal{N} = 2$ gauge theories can be found in [28]). The presence of the quartic coupling in this setup will be justified in a moment. θ is again given by eq. (2.3). At the special value $\psi = \frac{\pi}{2}$ the D5-branes are oriented along 012789 and we reproduce the configuration that gives the $\mathcal{N} = 2$ CS-SQCD theory without superpotential interactions.

For generic angle ψ the configuration (3.8) preserves $\mathcal{N} = 2$ supersymmetry in three dimensions. A quick verification of this fact appears in appendix A. The low-energy field theory exhibits $\mathcal{N} = 3$ supersymmetry enhancement for a special value of the quartic coupling, and therefore a special value of the angle ψ (see below). This effect is not visible in the brane configuration.

Before discussing further the low-energy gauge theory description of this setup it will be convenient to move the D5 and $(1,k)$ fivebranes along the x^6 direction past the NS5-brane. Once again, this motion leads to a Seiberg dual configuration of the form depicted in figure 1(b). At low energies the dynamics of this configuration is described by the magnetic version of $\mathcal{N} = 2$ CS-SQCD (level k and gauge group $\mathrm{U}(N_f + k - N_c)$) with a mass deformed superpotential

$$
\widetilde{W} = \sqrt{\mu} M_j^i q_i \tilde{q}^j + \frac{\alpha}{2} M_i^j M_j^i \ .
\tag{3.9}
$$

μ is a scale with the dimension of mass which was kept implicit before. The extra mass term for the meson M appears because of the rotation of the D5-branes. It captures the fact that the N_f D3-branes stretching between the D5-branes and the $(1,k)$ bound state can no longer move freely in the (89) plane. The mass parameter α is related to the rotation angle ψ via the relation

$$
\alpha = \mu \cot \psi \ .
\tag{3.10}
$$

By integrating out the massive fields M_j^i we obtain a superpotential with a quartic interaction for the magnetic quarks

$$
\widetilde{W} = -\frac{\tan \psi}{2} (q\tilde{q})^2 \ .
\tag{3.11}
$$

Returning to the electric description, we recognize that the deformation which is dual to M^2 is $(Q\widetilde{Q})^2$. Hence, in the presence of the rotated D5-branes the electric theory includes the quartic superpotential interaction

$$W = \frac{\beta^2 \cot \psi}{2} (Q\widetilde{Q})^2 \, , \tag{3.12}$$

where β is the proportionality constant that appears in the duality relation $M_j^i = \frac{\beta}{\sqrt{\mu}} Q^i \widetilde{Q}_j$. Requiring that the $\mathcal{N} = 3$ supersymmetry enhancement occurs simultaneously in the electric and magnetic theories gives

$$\tan \psi_{\mathcal{N}=3} = \frac{4\pi}{k} \, , \qquad \beta = \pm \frac{4\pi}{k} \, . \tag{3.13}$$

Again, we observe that Seiberg duality exchanges two versions of the same theory — the rank of the gauge group is dualized and the quartic superpotential coupling is essentially inversed.

3.3 Sextic deformations

The sextic operator $(Q\widetilde{Q})^3$ has classical dimension $\Delta_6 = 3$, and is therefore irrelevant at small coupling λ. As we increase the coupling for $x > 1$ the scaling dimension $\Delta_6 = 6R_Q(\lambda, x)$ goes down (presumably monotonically) until it reaches the minimum value $\frac{3}{2}$ at the supersymmetry breaking boundary $\frac{x}{x-1}$. This qualitative picture predicts that there is a critical coupling λ^* where the sextic operator becomes marginal. For this coupling

$$R_Q(\lambda^*, x) = \frac{1}{3} \, , \qquad x > 1 \, . \tag{3.14}$$

When the coupling lies in the range $\lambda^* < \lambda \leq \frac{x}{x-1}$ we can add this operator to the electric theory Lagrangian to generate an RG flow towards a new IR fixed point. Near the critical coupling λ^* the deformation δW_3 (see eq. (3.2)) is slightly relevant and an IR fixed point is expected to exist at perturbative values of α_3. It is nevertheless difficult to compute the β-function in conformal perturbation theory in this case since the theory is at finite coupling λ.

Applying Seiberg duality to the deformed theory we obtain a magnetic dual at level k with gauge group $\mathrm{U}(N_f + k - N_c)$ and superpotential

$$\widetilde{W} = \sqrt{\mu} M q\tilde{q} + \tilde{\alpha}_3 M^3 \, . \tag{3.15}$$

At weak magnetic coupling $\widetilde{\lambda} = 1 - \lambda\frac{x-1}{x} \ll 1$ the operator M^3 is relevant ($\Delta(M^3) \sim \frac{3}{2} < 2$) and the β-function is controlled to leading order in $\widetilde{\lambda}$ by a WZ model for the field M with cubic superpotential. This β-function is expected to have a zero at a finite value of $\tilde{\alpha}_3$.

4 CS-SQCD theories with one adjoint chiral superfield

4.1 Brief review of known results

In this section we will discuss the properties of a more complex set of theories. These are $\mathrm{U}(N_c)$ $\mathcal{N} = 2$ CSM theories at Chern-Simons level k coupled to N_f pairs of

(anti)fundamental chiral superfields Q^i, $\widetilde{Q}_i$ and one superfield X in the adjoint representation of the gauge group. In the absence of superpotential interactions we will call this theory the $\hat{\mathsf{A}}$ theory following a notation applied to analogous four-dimensional gauge theories in [27].

Applying the general arguments of [12], we deduce that the $\widehat{\mathsf{A}}$ theory is superconformal. Moreover, we will present a picture suggesting that, unlike the CS-SQCD theory without the adjoint superfield, in the $\widehat{\mathsf{A}}$ theory there is no range of parameters where supersymmetry is spontaneously broken. The global symmetry is $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f) \times \mathrm{U}(1)_a \times \mathrm{U}(1)_X \times \mathrm{U}(1)_R$.

The classical chiral ring consists of the single trace operators

$$\mathrm{Tr}\, X^{n+1}, \qquad\qquad n = 0, 1, \dots, \qquad\qquad (4.1)$$

made out of the adjoint chiral superfield X, and the generalized meson operators

$$Q^i X^n \widetilde{Q}_j, \qquad\qquad n = 0, 1, \dots\ . \qquad\qquad (4.2)$$

Since we consider theories with gauge group $\mathrm{U}(N_c)$ there are no baryon operators. Gauge invariant baryon-like operators constructed out of 't Hooft monopole operators [32] can exist though. We will not, however, consider such operators in this paper.

The addition of the superpotential interaction

$$W_{n+1} = \frac{g_0}{n+1} \mathrm{Tr}\, X^{n+1} \qquad\qquad (4.3)$$

to the Lagrangian truncates the above chiral ring to the finite subset

$$\mathrm{Tr}\, X^\ell, \qquad Q^i \widetilde{Q}_j, \qquad Q^i X^\ell \widetilde{Q}_j, \qquad \ell = 1, \dots, n-1\ . \qquad (4.4)$$

We will call the resulting 1-adjoint CS-SQCD theories A_{n+1}. The global symmetry of these theories is $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f) \times \mathrm{U}(1)_a \times \mathrm{U}(1)_R$. The $\mathrm{U}(1)_X$ symmetry has been broken explicitly by the superpotential W_{n+1}.

Some properties of the A_{n+1} theories, like Seiberg duality and spontaneous breaking of supersymmetry, can be deduced easily from a brane construction in string theory [26]. The relevant construction is a simple generalization of the type IIB string theory setup appearing in figure 1. It involves

$$
\begin{aligned}
n\ NS5 &: \quad 0\ 1\ 2\ 3\ 4\ 5 \\
1\ (1,k) &: \quad 0\ 1\ 2\ \begin{bmatrix} 3 \\ 7 \end{bmatrix}_\theta\ 8\ 9 \\
N_c\ D3 &: \quad 0\ 1\ 2\ |6| \\
N_f\ D5 &: \quad 0\ 1\ 2\ 7\ 8\ 9
\end{aligned}
\qquad (4.5)
$$

The new superpotential parameter n is encoded in the number of NS5-branes along 012345. For $n = 1$ the added superpotential is quadratic in the superfield X. Integrating out X we recover the CS-SQCD theories of section 2.

The rôle of the superpotential interaction $\mathrm{Tr}\, X^{n+1}$ in this setup can be identified in the following way. Displacing the n NS5-branes in the (89) plane to n different points

$a_\ell = x_\ell^8 + i x_\ell^9$ ($\ell = 1, 2, \ldots, n$) forces the N_c D3-branes to break up into n groups of r_1 D3-branes ending on the a_1 positioned NS5-brane, r_2 D3-branes ending on the a_2 positioned NS5-brane *etc.* with

$$\sum_{\ell=1}^{n} r_\ell = N_c . \tag{4.6}$$

The new configuration describes vacua of the gauge theory where the diagonal matrix elements of the complex scalar in the superfield X acquire expectation values a_i. These vacua are captured in gauge theory by a polynomial superpotential interaction of the form

$$W(X) = \sum_{\ell=0}^{n} \frac{g_\ell}{n+1-\ell} X^{n+1-\ell} . \tag{4.7}$$

For generic coefficients $\{g_\ell\}$ the superpotential has n distinct minima $\{a_\ell\}$ related to $\{g_\ell\}$ via the relation

$$W'(x) = \sum_{\ell=0}^{n} g_\ell x^{n-\ell} = g_0 \prod_{\ell=1}^{n} (x - a_\ell) . \tag{4.8}$$

In gauge theory the partition integers r_ℓ label the number of eigenvalues of the $N_c \times N_c$ matrix X residing in the ℓ-th minimum of the potential $V = |W'(x)|^2$. When all the expectation values are distinct the adjoint field is massive and the gauge group is Higgsed

$$\mathrm{U}(N_c) \to \mathrm{U}(r_1) \times \mathrm{U}(r_2) \times \cdots \times \mathrm{U}(r_n) . \tag{4.9}$$

In this vacuum we obtain n decoupled copies of the $\mathcal{N} = 2$ CS-SQCD theories at level k with N_f flavor multiplets.

The condition for the existence of a supersymmetric vacuum in the $\mathrm{U}(r_\ell)$ $\mathcal{N} = 2$ CS-SQCD theory is

$$r_\ell \leq k + N_f , \qquad 1 \leq \ell \leq n . \tag{4.10}$$

Summing over ℓ and sending all the a_i's to zero we obtain a condition for the existence of a supersymmetric vacuum in the A_{n+1} theory

$$N_c \leq n(k + N_f) . \tag{4.11}$$

The same condition can be obtained in string theory with the use of the *s*-rule of brane dynamics [26].

A Seiberg dual version of the A_{n+1} theory can be obtained by moving the D5 and $(1, k)$ fivebranes past the n NS5-branes along the x^6 direction as in figure 1(b). This duality, which was the subject of ref. [26], is a three-dimensional CS analog of Kutasov duality in four-dimensional $\mathcal{N} = 1$ gauge theory [33]. The magnetic A_{n+1} theory is an $\mathcal{N} = 2$ CSM theory at CS level k and gauge group $\mathrm{U}(n(N_f + k) - N_c)$ coupled to N_f pairs of chiral multiplets q_i, $\tilde{q}^i$, an adjoint chiral superfield Y and n magnetic mesons M_ℓ ($\ell = 1, \ldots, n$), each of which is an $N_f \times N_f$ matrix. There is a dual tree-level superpotential

$$\widetilde{W}_{n+1} = -\frac{g_0}{n+1} \mathrm{Tr}\, Y^{n+1} + \sum_{\ell=1}^{n} M_\ell \tilde{q} Y^{n-\ell} q . \tag{4.12}$$

All these statements have important consequences for the structure of the $\hat{\mathsf{A}}$ and A_{n+1} theories which we now proceed to uncover.

4.2 New results on R-charges

In the $\widehat{\mathsf{A}}$ theory the flavor symmetry $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f)$ guarantees that all the flavors Q^i, $\widetilde{Q}_i$ have the same $\mathrm{U}(1)_R$ charge $R_Q = R_Q(N_c, N_f, k)$. The $\mathrm{U}(1)_R$ charge of the superfield X is another function $R_X = R_X(N_c, N_f, k)$. We will not be able to say much about the function R_Q in this paper, but there is a number of interesting statements that we can make about the function R_X using information about the properties of the A_{n+1} theories.

The classical value of R_X in the $\widehat{\mathsf{A}}$ theory is $\frac{1}{2}$. Hence, at weak coupling, $\lambda = \frac{N_c}{k} \ll 1$, the chiral operators $\mathrm{Tr}\, X^2$, $\mathrm{Tr}\, X^3$ are relevant, the chiral operator $\mathrm{Tr}\, X^4$ is classically marginal and the chiral operators $\mathrm{Tr}\, X^n$ ($n > 4$) are irrelevant. Therefore, adding $\mathrm{Tr}\, X^n$ ($n > 4$) to the Lagrangian at weak coupling will not modify the IR behavior of the theory.

The fact that supersymmetry can be spontaneously broken in the A_{n+1} theory proves that as we increase the coupling λ, R_X receives in the $\widehat{\mathsf{A}}$ theory large negative anomalous contributions which eventually make the operator $\mathrm{Tr}\, X^{n+1}$ relevant. At the supersymmetry breaking value of λ the operator $\mathrm{Tr}\, X^{n+1}$ modifies the IR behavior of the theory so drastically that the supersymmetric vacuum is lifted.[4] Presicely how R_X decreases and whether or not an operator $\mathrm{Tr}\, X^{n+1}$ can become relevant in the $\widehat{\mathsf{A}}$ theory depends on the value of x, *i.e.* the ratio N_c/N_f.

In terms of the parameters λ and x a supersymmetric vacuum in the A_{n+1} theory exists for all λ if $x \leq n$, and for λ such that

$$\lambda \leq \lambda_{n+1}^{\mathrm{SUSY}} = \frac{nx}{x - n} \tag{4.13}$$

for $x > n$.

Now let us fix the value of x and discuss what happens in the $\hat{\mathsf{A}}$ theory as we increase the coupling λ. At small λ and large x a perturbative calculation based on the results of [12] gives

$$R_X(\lambda, x) \sim \frac{1}{2} - \left(2 + \frac{4}{3x}\right)\lambda^2 + \cdots, \tag{4.14}$$

where the dots $\cdots$ denote subleading contributions. As we further increase λ the R-charge R_X continues to decrease. For all integers $n \leq [x]$ ($[x]$ denotes the integer part of x) there is a sequence of critical values

$$0 = \lambda_2^* = \lambda_3^* = \lambda_4^* < \lambda_5^* < \cdots < \lambda_n^* < \lambda_{n+1}^* < \cdots < \lambda_{[x]}^* < \lambda_{[x]+1}^* \tag{4.15}$$

where each time one of the chiral operators $\mathrm{Tr}\, X^{n+1}$ ($n \leq [x]$) becomes marginal. By definition, λ_{n+1}^* is the point where the chiral operator $\mathrm{Tr}\, X^{n+1}$ becomes marginal, *i.e.* the point where

$$R_X(\lambda_{n+1}^*, x) = \frac{2}{n+1} . \tag{4.16}$$

Above λ_{n+1}^* the operator $\mathrm{Tr}\, X^{n+1}$ is relevant. Adding it to the Lagrangian in this range of parameters drives the theory to a new fixed point — the A_{n+1} theory. By further

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[4]A similar observation relating the anomalous dimensions of X and vacuum stability can also be found in refs. [34, 35] in the context of four-dimensional $\mathcal{N} = 1$ adjoint SQCD theories. I thank David Kutasov for a discussion that prompted me to think more about this relation.
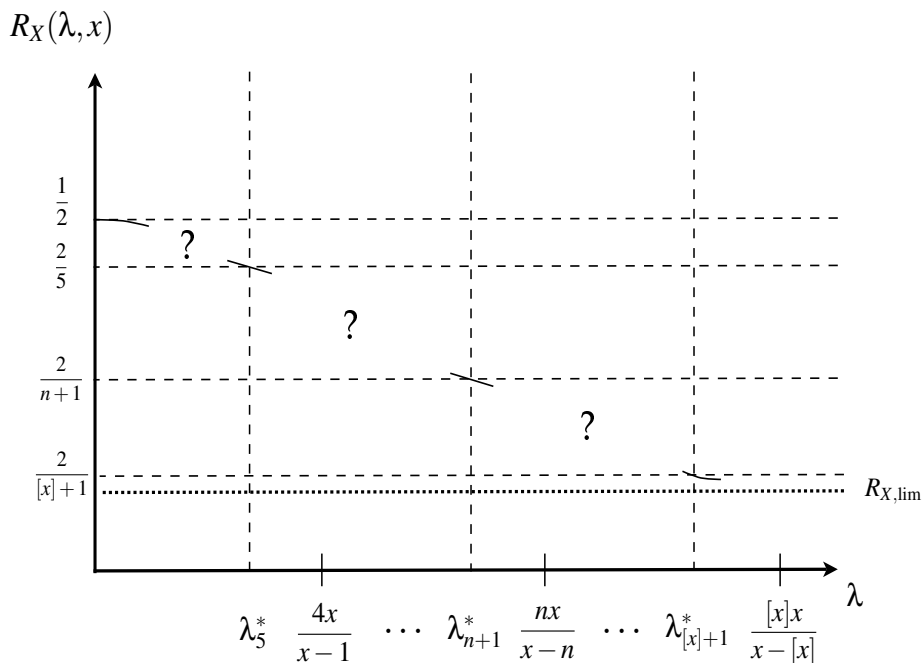
Figure 4. A plot of R_X in the $\hat{\mathsf{A}}$ theory as a function of λ for fixed x. R_X interpolates between its classical value $\frac{1}{2}$ and a limiting value $R_{X,\mathrm{lim}}$ in the range $\frac{1}{2([x]+2)} < R_{X,\mathrm{lim}} < \frac{2}{[x]+1}$. λ_{n+1}^* is the point where the chiral operator $\mathrm{Tr}\,X^{n+1}$ becomes marginal. The critical point $\frac{nx}{x-n}$ is where the A_{n+1} theory exhibits spontaneous breaking of supersymmetry.

increasing the coupling in the A_{n+1} theory we reach the SUSY breaking point $\lambda_{n+1}^{\mathrm{SUSY}}$ where the supersymmetric vacuum is destabilized. This observation provides an upper bound on the exact value of λ_{n+1}^*

$$\lambda_{n+1}^* < \lambda_{n+1}^{\mathrm{SUSY}} = \frac{nx}{x-n} \ . \tag{4.17}$$

The emerging picture is depicted graphically in figure 4. The R-charge R_X decreases monotonically as we increase λ making more and more single trace chiral operators $\mathrm{Tr}\,X^{n+1}$ relevant. Beyond the critical coupling $\lambda_{[x]+1}^*$, R_X approaches a limiting lowest value $R_{X,\mathrm{lim}} > 0$, which lies somewhere inside the interval $\left(\frac{1}{2([x]+2)}, \frac{2}{[x]+1}\right)$. The lower bound of this interval arises in the following way.

The scaling dimension of the operator $\mathrm{Tr}\,X^{[x]+2}$ is $\Delta_{[x]+2} = ([x]+2)R_X(\lambda, x)$. We cannot exclude the possibility that this operator becomes relevant beyond some coupling, but even if it does it cannot destabilize the supersymmetric vacuum in the $\mathsf{A}_{[x]+2}$ theory. If the function $R_X(\lambda, x)$ continues to decrease monotonically towards zero, a value of λ will be reached eventually where $\Delta_{[x]+2} = \frac{1}{2}$. Beyond this point the operator $\mathrm{Tr}\,X^{[x]+2}$ becomes a free field and decouples from the rest of the theory. Hence, in this regime, a deformation by a superpotential interaction linear in $\mathrm{Tr}\,X^{[x]+2}$ will break the supersymmetry, something that we know from the above analysis cannot happen. We conclude that $\Delta_{[x]+2} > \frac{1}{2}$ for all λ, which implies $R_{X,\mathrm{lim}} > \frac{1}{2([x]+2)}$.

A qualitatively similar situation occurs in the four-dimensional analog of this theory — the IR of the $\mathcal{N} = 1$ adjoint SQCD theory — as we vary the single parameter x. In that case, we can compute exactly where the critical values of x lie using a-maximization [35].

In our case, it would be nice to know how fast an operator $\mathrm{Tr}\, X^{n+1}$ becomes relevant. In other words, it would be nice to have an estimate of the magnitude of the difference $\lambda_{n+1}^{\mathrm{SUSY}} - \lambda_{n+1}^*$. Fortunately, such an estimate is within the power of our current considerations.

We have observed that as we increase λ the scaling dimension of the generic chiral operator $\mathrm{Tr}\, X^{n+1}$, $\Delta_{n+1} = (n+1)R_X(\lambda, x)$, decreases. If Δ_{n+1} reaches the unitarity bound $\frac{1}{2}$ at some value of λ, call it $\lambda_{n+1}^{\mathrm{max}}$, then beyond this point the operator $\mathrm{Tr}\, X^{n+1}$ becomes free and decouples from the rest of the theory. For reasons similar to the ones outlined above this cannot happen before we reach the SUSY breaking point $\lambda_{n+1}^{\mathrm{SUSY}}$ of the A_{n+1} theory. We can determine $\lambda_{n+1}^{\mathrm{max}}$ in the following way.

As we increase λ beyond λ_{n+1}^*, we reach the critical coupling $\lambda_{n'+1}^*$ ($n' > n$) of another chiral operator $\mathrm{Tr}\, X^{n'+1}$. There is an integer n' for which $\mathrm{Tr}\, X^{n'+1}$ is marginal and simultaneously $\mathrm{Tr}\, X^{n+1}$ becomes free. This occurs when

$$\Delta_{n+1} = (n+1)R_X(\lambda_{n'+1}^*, x) = \frac{2(n+1)}{n'+1} = \frac{1}{2} \quad \Leftrightarrow \quad n' = 4n+3 \ . \tag{4.18}$$

This, of course, will be true as long as $n' \leq [x]$, i.e. $n \leq \frac{[x]-3}{4}$. Assuming this inequality, we deduce that

$$\lambda_{n+1}^{\mathrm{max}} = \lambda_{n'+1}^* = \lambda_{4(n+1)}^* \tag{4.19}$$

and our previous observations imply

$$\lambda_{n+1}^* < \frac{nx}{x-n} < \lambda_{n+1}^{\mathrm{max}} = \lambda_{4(n+1)}^* \ . \tag{4.20}$$

The second inequality provides a lower bound to λ_{n+1}^*

$$\frac{\left[\frac{n-3}{4}\right]x}{x - \left[\frac{n-3}{4}\right]} < \lambda_{n+1}^* \tag{4.21}$$

and gives an estimate to the difference $\lambda_{n+1}^{\mathrm{SUSY}} - \lambda_{n+1}^*$ provided $n \leq \frac{[x]-3}{4}$. At large values of x, i.e. when $N_f \ll N_c$, the combination of the lower and upper bounds (4.21) and (4.17) gives the inequalities

$$\left[\frac{n-3}{4}\right] < \lambda_{n+1}^* < n \ . \tag{4.22}$$

4.3 More on Seiberg duality

Let us denote compactly as $\widetilde{\widetilde{\mathsf{A}}}$ the set of $\mathrm{U}(N_c)$ $\mathcal{N} = 2$ CSM theories at CS level k without superpotential interactions that are coupled to N_f quark multiplets q_i, $\tilde{q}^i$, an adjoint chiral superfield Y and n gauge singlet superfields M_ℓ ($\ell = 1, 2, \ldots, n$), each of which is an $N_f \times N_f$ matrix. The magnetic description of the $\mathrm{U}(N_c)$ A_{n+1} theory arises from the $\mathrm{U}(n(N_f + k) - N_c)$ $\widetilde{\widetilde{\mathsf{A}}}$ theory after the superpotential interaction (4.12) is added to the Lagrangian.

The U$(n(N_f + k) - N_c)$ $\widetilde{\widetilde{\mathsf{A}}}$ theories are superconformal field theories with large-N parameters

$$\widetilde{\lambda} = \frac{n(N_f + k) - N_c}{k} = n - \lambda\left(1 - \frac{n}{x}\right) , \quad \widetilde{x} = \frac{n(N_f + k) - N_c}{N_f} = n + x\left(\frac{n}{\lambda} - 1\right) . \quad (4.23)$$

They are weakly coupled when $\widetilde{\lambda} \ll 1$. Assuming $n > 3$, all the operators appearing in $\widetilde{W}_{n+1}$ (eq. (4.12)) are irrelevant in the perturbative regime, except for the cubic operator $M_n\tilde{q}q$ and the quartic $M_{n-1}\tilde{q}Yq$. Both of them are relevant (the quartic operator is classically marginal with perturbatively negative anomalous dimension). Adding the operators $M_n\tilde{q}q$ and $M_{n-1}\tilde{q}Yq$ to the Lagrangian as superpotential interactions drives the theory to a new interacting fixed point.

More and more terms in the superpotential (4.12) are expected to become relevant in the $\widetilde{\widetilde{\mathsf{A}}}$ theory as we increase $\widetilde{\lambda}$. Notice that the elementary and composite mesons (M_ℓ and $\tilde{q}Y^{n-\ell}q$ respectively) are Legendre-transform conjugate variables in the magnetic theory. Therefore, depending on whether the term $M_\ell\tilde{q}Y^{n-\ell}q$ is relevant or not in the magnetic superpotential, we should include either M_ℓ or $\tilde{q}Y^{n-\ell}q$ in the spectrum of independent operators.

Ultimately, as we increase $\widetilde{\lambda}$ we should encounter a critical coupling $\widetilde{\lambda}^*_{n+1}$ above which the operator $\mathrm{Tr}\, Y^{n+1}$ is relevant and both the electric and magnetic theories flow towards the A_{n+1} fixed point. We can write this critical coupling as

$$\widetilde{\lambda}^*_{n+1} = n - \lambda^{**}_{n+1}\left(1 - \frac{n}{x}\right) \quad (4.24)$$

In terms of the electric 't Hooft coupling the magnetic theory enters the phase with $\widetilde{\lambda} > \widetilde{\lambda}^*_{n+1}$ when

$$\lambda\left(-1 + \frac{n}{x}\right) > \lambda^{**}_{n+1}\left(-1 + \frac{n}{x}\right) . \quad (4.25)$$

Demanding that the A_{n+1} fixed point can be obtained simultaneously by adding the relevant operator $\mathrm{Tr}\, X^{n+1}$ to the electric theory means $x > n$ and $\lambda > \lambda^*_{n+1}$. All these conditions can be met if and only if

$$\lambda^*_{n+1} < \lambda^{**}_{n+1} < \lambda^{\mathrm{SUSY}}_{n+1} . \quad (4.26)$$

Verifying this prediction requires a strong analytic tool — the analog of a-maximization in four dimensions.

Inside the 'window' $[\lambda^*_{n+1}, \lambda^{**}_{n+1}]$ both the electric and magnetic theories flow to the same IR fixed point where

$$R_X = R_Y = \frac{2}{n+1} . \quad (4.27)$$

From the remaining R-charges (R_Q for Q, $\widetilde{Q}$, R_q for q, $\tilde{q}$, and R_{M_ℓ} for M_ℓ) only one is independent. The map between electric and magnetic meson fields and the marginality of the mesonic superpotential interactions implies the relations

$$R_Q + R_q = \frac{2}{n+1} , \qquad R_{M_\ell} = \frac{2(\ell - 1)}{n+1} + 2R_Q , \qquad \ell = 1, 2, \ldots, n . \quad (4.28)$$

Finally, matching the chiral rings and mapping superpotential deformations of the electric A_{n+1} theory to its magnetic dual is something that can be achieved precisely as in four dimensions [36]. The classical chiral rings do not match, but the quantum chiral rings do. Since the adjoint field X in the electric theory is an $N_c \times N_c$ matrix it obeys automatically, by virtue of the Caley-Hamilton theorem, the restrictions that follow from the characteristic equation

$$f(X) = 0, \quad \text{where} \quad f(z) \equiv \det(z - X) . \tag{4.29}$$

To obtain the quantum chiral ring, the classical chiral ring relations must be supplemented by the characteristic equation of the magnetic theory. Analogous statements apply to the magnetic theory.

Mapping superpotential deformations of the form (4.7) under Seiberg duality entails the steps taken in the four-dimensional adjoint SQCD theories in [36]. A minor difference with the analysis of [36] arises from the fact that here we discuss $\mathrm{U}(N_c)$, instead of $\mathrm{SU}(N_c)$, gauge groups.

4.4 A special case and comments on holography

Many supersymmetric CSM theories, with prototype the $\mathcal{N} = 6$ CSM theories in [13], admit a holographic dual description in terms of either string theory or M-theory on some AdS_4 background of the form $AdS_4 \times \mathcal{M}$, with $\mathcal{M}$ being some compact manifold. One may wonder whether the $\mathcal{N} = 2$ CSM theories in this section have a similar holographic AdS_4 description.

A special case without the usual complications of fundamental matter is the case of the $\mathrm{U}(N_c)$ 1-adjoint CS-SQCD theories with $N_f = 0$. Besides the superconformal fixed points $\widehat{A}$, labeled by the integers N_c, k, new fixed points A_{n+1} can be obtained by adding the superpotential interactions

$$W_{n+1} = \frac{g_0}{n+1} \operatorname{Tr} X^{n+1} \tag{4.30}$$

for any integer $n \geq 1$ and λ greater than a critical value λ_{n+1}^*. Each of the $\mathrm{U}(N_c)$ A_{n+1} theories admits a Seiberg dual description in terms of another A_{n+1} theory at the same level k but different gauge group $\mathrm{U}(nk - N_c)$. The dual description disappears when the supersymmetric vacuum is spontaneously broken in the original theory. The condition for the existence of a supersymmetric vacuum in the $\mathrm{U}(N_c)$ theory is

$$N_c \leq nk \quad \Leftrightarrow \quad \lambda \leq n . \tag{4.31}$$

There is a regime, $\lambda \in [\lambda_{n+1}^{**}, n]$ in the notation of the previous subsection, where the operator $\operatorname{Tr} X^{n+1}$ is an irrelevant operator in the dual description. In that case, Seiberg duality exchanges an A_{n+1} fixed point with an $\widehat{A}$ fixed point.

The standard large-N reasoning suggests that these superconformal field theories have a holographic string theory description. Symmetries imply that this description involves non-critical strings on some $AdS_4 \times S^1$ background, presumably with curvature of order the string scale. The symmetries of AdS_4 reproduce the field theory superconformal symmetries and the S^1 the internal $\mathrm{U}(1)_R$ symmetry.

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The possibility of a holographic description for the $\widehat{\mathsf{A}}$ fixed points was also discussed in [12]. The setup in [12] involves N_c M5-branes wrapping a special Lagrangian Lens space $S^3/\mathbb{Z}_k$ in a Calabi-Yau three-fold. There is no AdS_4 solution for this system in supergravity which implies that α' corrections are indeed important.

The A_{n+1} theories are also related to wrapped M5-branes. For example, we can realize the A_2 theory with finite coupling $g_0 = -\frac{k}{4\pi}$ as a special case of the configuration (3.6) with $N_f = 0$. This setup is a special case of the configurations analyzed in ref. [17]. Compactifying the x^6 direction, T-dualizing, and lifting to M-theory converts the suspended D3-branes into 'fractional M2-branes', $i.e.$ M5-branes wrapping a vanishing 3-cycle at a $\mathbb{Z}_k$ orbifold point.

5 Two-adjoint theories: RG flows from the $\widehat{\mathsf{O}}$ theory

So far we have discussed $\mathcal{N} = 2$ CSM theories with an arbitrary number of fundamental/anti-fundamental pairs of chiral multiplets and one adjoint chiral multiplet. These theories comprise a small set in the larger domain of $\mathcal{N} = 2$ theories with two, instead of one, chiral superfields. In this section, we will explore RG flows and fixed points in this wider setup. The zoo of $\mathcal{N} = 1$ superconformal field theories with two adjoint chiral superfields in four dimensions was discussed, using a-maximization techniques, in ref. [27]. In that work an intriguing ADE classification of fixed points was observed. We will find that a subset of our three-dimensional superconformal field theories admits a similar classification.

5.1 Relevant deformations to $\widehat{\mathsf{A}}$, $\widehat{\mathsf{D}}$, $\widehat{\mathsf{E}}$

Our starting point is a theory which, mimicking [27], we will call the $\widehat{\mathsf{O}}$ theory. By definition, this theory is a U(N_c) $\mathcal{N} = 2$ CSM theory at level k coupled to N_f pairs of fundamental/anti-fundamental chiral superfields and two chiral superfields in the adjoint representation. We will denote the adjoint chiral superfields X and X'. The $\widehat{\mathsf{O}}$ theory has no superpotential interactions. It is superconformal by the arguments of ref. [12] and possesses the global symmetry group $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f) \times \mathrm{SU}(2) \times \mathrm{U}(1)_X \times \mathrm{U}(1)_R$. The SU(2) symmetry rotates the (X, X') doublet. Because of this symmetry the $\mathrm{U}(1)_R$ charges of X and X' are identical and will be denoted by R_X, which is a function of the parameters k, N_c, N_f.

The chiral ring includes single trace operators of the form $\mathrm{Tr}\, X^n X'^{n'}$ $(n, n' = 0, 1, \ldots)$ up to arbitrary permutations of the fields X, X' inside the trace. The scaling dimension of such operators is

$$(n + n')R_X(N_c, N_f, k) \ . \tag{5.1}$$

Hence, any of the (n, n') operators is relevant when

$$R_X(N_c, N_f, k) < \frac{2}{n + n'} \ . \tag{5.2}$$

In that case, we can add the operator to the $\widehat{\mathsf{O}}$ Lagrangian as a superpotential interaction to generate an RG flow towards a new IR fixed point.

As in the analysis of the previous sections we expect the function R_X to decrease as λ becomes larger and larger and the gauge interactions stronger. This can be verified explicitly with a two-loop calculation in the perturbative regime [12]

$$R_X(\lambda, x) \sim \frac{1}{2} - \left(3 + \frac{4}{3x}\right)\lambda^2 + \cdots . \tag{5.3}$$

Since we have very limited information about the non-perturbative behavior of the function R_X, we will concentrate, in what follows, to operators that are either already classically relevant or classically marginal but quantum mechanically relevant.

Deformations involving only the operators $\operatorname{Tr} X$, $\operatorname{Tr} X'$ will not be considered since they lead to F-term equations that cannot be solved. We will consider the following (inequivalent) quadratic and cubic superpotential deformations:

(1) $W = \operatorname{Tr} XX'$. In this case, both chiral superfields X, X' are massive and the RG flow interpolates between the $\widehat{\mathsf{O}}$ theory and CS-SQCD with N_f flavors.

(2) $W = \operatorname{Tr} X'^2$. The chiral superfield X' is massive and can be integrated out. The IR fixed point is the $\widehat{\mathsf{A}}$ theory that was analyzed in the previous section.

(3) $W = \operatorname{Tr} XX'^2$. The RG flow leads to a new IR fixed point, which we will call the $\widehat{\mathsf{D}}$ theory.

(4) $W = \operatorname{Tr} X'^3$. The IR fixed point arising from this RG flow will be named $\widehat{\mathsf{E}}$.

The (inequivalent) quartic deformations $W = \operatorname{Tr} X^4, \operatorname{Tr} X^3 X', \operatorname{Tr} X^2 X'^2, \operatorname{Tr} XX'XX'$ are marginal at $\lambda = 0$ but relevant at $\lambda > 0$. The generated RG flows are a special case of the flows studied in perturbation theory in [12]. We will not have anything new to add concerning this case. Instead, we will proceed to examine RG flows away from the $\widehat{\mathsf{A}}$, $\widehat{\mathsf{D}}$, $\widehat{\mathsf{E}}$ theories which have certain similarities with RG flows in two-dimensional $\mathcal{N} = (2,2)$ Landau-Ginzburg models and four-dimensional $\mathcal{N} = 1$ SQCD theories with two adjoints.

5.2   Flows from $\widehat{\mathsf{A}} \to \mathsf{A}_{n+1}$

RG flows from the $\widehat{\mathsf{A}}$ theory to A_{n+1} fixed points occur when the superpotential $W = \operatorname{Tr} X^{n+1}$ is relevant. The conditions for this to happen were explored in the previous section. We have not shown explicitly that the IR of this RG flow is indeed a fixed point of the β-function. This is a hard question because the A_{n+1} theory is non-perturbative for generic n. The only exception is the case $n = 3$, i.e. the case of a quartic deformation. Here one can show the existence of a perturbative fixed point with a two-loop computation of the β-function [12].

5.3   Flows from $\widehat{\mathsf{D}} \to \mathsf{D}_{n+2}$

The $\widehat{\mathsf{D}}$ theory arises from the $\widehat{\mathsf{O}}$ theory after the deformation by the superpotential interaction $W_{\widehat{\mathsf{D}}} = \operatorname{Tr} XX'^2$. In this theory the chiral ring of gauge invariant operators is subject to the relations coming from the $W_{\widehat{\mathsf{D}}}$ equations of motion

$$\partial_{X'} W_{\widehat{\mathsf{D}}} = \{X, X'\} = 0, \qquad \partial_X W_{\widehat{\mathsf{D}}} = X'^2 = 0 . \tag{5.4}$$

The first equation is particularly convenient, because we can use it to freely re-order the fields X, X' inside traces (up to a minus sign). Using these equations we find that the chiral ring is generated by the single trace operators

$$\mathrm{Tr}\, X^\ell\,, \qquad \ell \geq 1\,, \qquad \mathrm{Tr}\, X' \tag{5.5}$$

and the meson operators

$$M_{\ell,s} = \tilde{Q} X^\ell X'^s Q\,, \qquad \ell \geq 0\,, s = 0,1\,. \tag{5.6}$$

Note that $\mathrm{Tr}\, X^n X' = 0$ because of the first equation in (5.4) and the cyclicity of the trace.

In the $\widehat{\mathsf{D}}$ theory the superpotential interaction $W_{\widehat{\mathsf{D}}}$ is marginal and imposes the following constraint on the R-charges

$$R_X + 2R_{X'} = 2\,. \tag{5.7}$$

Hence, the independent R-charges that need to be determined as functions of the parameters k, N_c, N_f are the R-charge R_X of X and the common R-charge R_Q of the (anti)fundamental multiplets Q, $\widetilde{Q}$.

Any of the chiral operators (5.5), (5.6) can be used to deform the Lagrangian of the $\widehat{\mathsf{D}}$ theory. We will focus on superpotential deformations involving the first set (5.5). Without prior knowledge of the R-charges R_X, $R_{X'}$ it is unclear which of these deformations are relevant and if so for what range of parameters. If this case is similar to the $\widehat{\mathsf{A}}$ theory we might expect that R_X decreases as we increase λ and, because of eq. (5.7), at the same time $R_{X'}$ increases. Assuming this is true, and that there is a range of parameters where the operator $\mathrm{Tr}\, X^{n+1}$ is relevant, we can deform by the superpotential interaction $\Delta W = \mathrm{Tr}\, X^{n+1}$ to flow towards a tentative new fixed point which we will call D_{n+2}. In what follows, we will argue in favor of these flows and will propose that the D_{n+2} fixed points exist and exhibit non-trivial properties, among them Seiberg duality.

Before proceeding further, notice that the $n = 1$ deformation involves the superpotential

$$W_{n=1} = \frac{g}{2} \mathrm{Tr}\, X^2 + a\, \mathrm{Tr}\, X X'^2\,. \tag{5.8}$$

The field X is massive in this case and by integrating it out we get the low energy superpotential

$$W = -\frac{a^2}{2g} \mathrm{Tr}\, X'^4\,. \tag{5.9}$$

In this way, we recover the 1-adjoint CS-SQCD fixed point A_4.

5.3.1 Stability bounds and their consequences

The crucial element that allowed us in the $\widehat{\mathsf{A}}$ theories to determine the qualitative behavior of the R-charge R_X was a bound on λ for the stability of the supersymmetric vacuum in the A_{n+1} theories. We could read these bounds directly from the s-rule in a string theory setup, or by deforming slightly the superpotential (as in eq. (4.7)), flowing to a product of CS-SQCD vacua and then using the condition for the existence of a supersymmetric vacuum in CS-SQCD. We can repeat the second argument in the D_{n+2} field theories. A

similar analysis was performed in ref. [27] for the four-dimensional D_{n+2} theories. Since the argument is identical in our case we will focus on the main points highlighting elements particular to our case and defer the reader to [27] for additional details.

The core of the argument centers around the precise way in which the chiral ring truncates in the presence of the D_{n+2} superpotential

$$W_{\mathsf{D}_{n+2}} = \operatorname{Tr} X^{n+1} + \operatorname{Tr} X X'^2 \ . \tag{5.10}$$

For simplicity, we keep the superpotential coefficients in front of each term implicit. The relations coming from this superpotential are

$$\{X, X'\} = 0 \,, \qquad\qquad X^n + X'^2 = 0 \ . \tag{5.11}$$

What happens to the chiral ring depends crucially on whether n is odd or even.

For odd n there is a drastic truncation of the chiral ring. Using the relations (5.11) one can show that $X'^3 = 0$. The classical chiral ring includes the single trace operators

$$\operatorname{Tr} X^{\ell-1} \,, \quad \ell = 1, \ldots, n \,, \quad \operatorname{Tr} X' \,, \quad \operatorname{Tr} X'^2 \,, \quad \operatorname{Tr} X^{2m} X'^2 \,, \quad m = 1, \ldots, \frac{n-1}{2} \,, \tag{5.12}$$

where the order of X and X' does not matter because of the first relation in (5.11), and the $3nN_f^2$ mesons

$$\mathcal{M}_{\ell,s} = \widetilde{Q} X^{\ell-1} X'^{s-1} Q \,, \qquad \ell = 1, \ldots, n \,, \qquad s = 1, 2, 3 \ . \tag{5.13}$$

To determine the conditions for the existence of a supersymmetric vacuum we deform the superpotential $W_{\mathsf{D}_{n+2}}$ by lower order terms

$$W = \operatorname{Tr} \left(F_{n+1}(X) + X X'^2 + X' \right) \tag{5.14}$$

where $F_{n+1}(X)$ is a degree $n+1$ polynomial in X. The F-term equations for this superpotential are

$$\{X, X'\} = -1 \,, \qquad\qquad X'^2 + \partial_X F_{n+1}(X) = 0 \ . \tag{5.15}$$

The vacua of the field theory are solutions of these equations. The irreducible representations of the algebra defined by (5.15) are $n+2$ different one-dimensional representations and $\frac{n-1}{2}$ two-dimensional representations [27, 37]. The general vacuum has r_a copies of the a-th one-dimensional representation ($a = 1, \ldots, n+2$), and s_b copies of the b-th two-dimensional representation ($b = 1, \ldots, \frac{n-1}{2}$). In such a vacuum the gauge group is Higgsed to

$$\mathrm{U}(N_c) \rightarrow \prod_{a=1}^{n+2} \mathrm{U}(r_a) \prod_{b=1}^{\frac{n-1}{2}} \mathrm{U}(s_b) \quad \text{with} \quad \sum_{a=1}^{n+2} r_a + \sum_{b=1}^{\frac{n-1}{2}} 2 s_b = N_c \tag{5.16}$$

and both adjoint chiral superfields are massive. Each $\mathrm{U}(r_a)$ factor has N_f pairs of flavor multiplets and each $\mathrm{U}(s_b)$ factor $2N_f$ flavor pairs [27]. Hence, each factor is a CS-SQCD theory with some number of flavor multiplets (N_f or $2N_f$). The condition for the existence of a supersymmetric vacuum in CS-SQCD (see eq. (2.4)) implies for the product (5.16)

$$r_a \leq N_f + k \,, \qquad s_b \leq 2(N_f + k) \,, \qquad a = 1, \ldots, n+2 \,, \qquad b = 1, \ldots, \frac{n-1}{2} \ . \tag{5.17}$$

Summing up these inequalities we obtain a condition for the existence of a supersymmetric vacuum in the D_{n+2} theory

$$N_c \leq 3n(N_f + k) . \tag{5.18}$$

Notice that for $n = 1$ this formula reproduces the condition (4.11) for the A_4 theory which is consistent with the observation in eq. (5.9) that D_3 is essentially the A_4 theory.

For even n the situation is more complex. In this case, the classical chiral ring does not truncate and a bound like (5.18) cannot be derived classically. However, a bound may exist at the quantum level. The fact that we can add a superpotential deformation $\Delta W = \text{Tr}\, X^{n'+1}$ to the D_{n+2} theory with $n' < n$, n odd and n' even, to flow from the D_{n+2} theory to the $\mathsf{D}_{n'+2}$ theory suggests that the even n theories also have a stability bound. This is based on the natural expectation that by adding a more relevant term to the superpotential it will be easier for the vacuum to get destabilized. A natural hypothesis is that the stability bound for n even continues to obey the same form that was found in eq. (5.18). Further motivation for this hypothesis will be provided in a moment. In four-dimensional $\mathcal{N} = 1$ adjoint SQCD theories of the D_{n+2} type this assumption is corroborated by the a-conjecture [27].

Assuming the validity of (5.18) for all n as a working hypothesis, we can deduce a qualitative picture for the λ-dependence of the R-charge R_X in the $\widehat{\mathsf{D}}$ theory, which is similar to that in the 1-adjoint $\widehat{\mathsf{A}}$ theory. In terms of the 't Hooft parameters λ, x a supersymmetric vacuum exists in the D_{n+2} theory for all λ if $x \leq 3n$, and for λ such that

$$\lambda \leq \lambda_{n+2}^{\text{SUSY}} = \frac{3nx}{x - 3n} \tag{5.19}$$

for $x > 3n$. Hence, as we increase λ in the $\widehat{\mathsf{D}}$ theory the R-charge R_X decreases and for $n < \frac{x}{3}$ the operator $\text{Tr}\, X^{n+1}$ becomes marginal at a critical coupling

$$\lambda_{n+2}^* < \lambda_{n+2}^{\text{SUSY}} = \frac{3nx}{x - 3n} . \tag{5.20}$$

The qualitative behavior of R_X is the same as in figure 4 with the obvious modifications. The analog of the inequalities (4.22) is

$$3 \left[\frac{n-3}{4} \right] < \lambda_{n+2}^* < 3n . \tag{5.21}$$

The behavior of $R_{X'}$ is fixed in terms of the relation (5.7).

5.3.2 Seiberg-Brodie duality

It has been argued [38] that the four-dimensional D_{n+2} theories exhibit Seiberg duality. It is tempting to propose that the D_{n+2} CSM theories in this section also exhibit Seiberg duality augmenting the known list of $\mathcal{N} = 2$ Chern-Simons theories with Seiberg duals.

Previous experience with Seiberg duality in $\mathcal{N} = 2$ CSM theories [25, 26] shows that the rank of the dual gauge group encodes the spontaneous supersymmetry breaking boundary of the original theory in a simple fashion. Extending this feature to the D_{n+2} theories we propose that they have a dual magnetic description in terms of an $\mathcal{N} = 2$ CSM theory at the

same level k and gauge group $\mathrm{U}(3n(N_f+k)-N_c)$.[5] The matter content and the superpotential interactions of the dual magnetic theory are partially fixed by the chiral ring structure. The details work as in four dimensions, hence we propose that the matter content of the dual theory includes N_f pairs of (anti)fundamental multiplets q_i, $\tilde{q}^i$, two adjoint chiral superfields Y, Y' and $3nN_f^2$ gauge singlets $(M_{\ell,s})^i_j$ ($\ell = 1, \ldots, n$, $s = 1, 2, 3$, $i, j = 1, \ldots, N_f$), which are the magnetic duals of the meson operators (5.13). The dual tree-level superpotential is

$$\widetilde{W}_{\mathsf{D}_{n+2}} = \mathrm{Tr}\, Y^{n+1} + \mathrm{Tr}\, YY'^2 + \sum_{\ell=1}^{n}\sum_{s=1}^{3} M_{\ell,s}\, \tilde{q}\, Y^{n-\ell} Y'^{3-s} q \; . \tag{5.22}$$

Weak evidence for the validity of this duality is presented in appendix B.

When $3n < x$ this duality works as a strong/weak duality. The magnetic 't Hooft coupling $\widetilde{\lambda}$ is related to the electric coupling λ in the following way

$$\widetilde{\lambda} = 3n - \left(1 - \frac{3n}{x}\right)\lambda \; . \tag{5.23}$$

Repeating the discussion of section 4.3 we anticipate a window $[\lambda_{n+2}^*, \lambda_{n+2}^{**}]$ inside which both $\mathrm{Tr}\, X^{n+1}$ and its dual $\mathrm{Tr}\, Y^{n+1}$ are relevant operators.

5.4 Flows from $\widehat{\mathsf{E}}$

The $\widehat{\mathsf{E}}$ theory arises from the $\widehat{\mathsf{O}}$ theory after the superpotential deformation

$$W_{\widehat{\mathsf{E}}} = \mathrm{Tr}\, X'^3 \; . \tag{5.24}$$

The equations of motion for this superpotential impose the classical chiral ring relation $X'^2 = 0$ and truncate the chiral ring of the $\widehat{\mathsf{O}}$ theory to the operators

$$\mathrm{Tr}\, X_{i_1}\cdots X_{i_n}, \quad \widetilde{Q}Q, \quad \widetilde{Q}X_{i_1}\cdots X_{i_n}Q, \ \text{with} \ i_1, i_2\cdots = 1, 2, \quad X_1 = X\,,\, X_2 = X'\,, \quad n = 1, 2, \ldots \tag{5.25}$$

with the provision that there are no adjacent X' operators in the above combinations (including adjacency via cyclic permutation).

In the $\widehat{\mathsf{E}}$ theory the R-charge of the X' field is fixed by the superpotential

$$R_{X'} = \frac{2}{3} \; . \tag{5.26}$$

What remains to be computed are the common R-charge R_Q of the quarks Q, $\widetilde{Q}$, and the R-charge R_X of the adjoint field X.

Since we lack an exact analytic tool that allows us to compute these charges we will restrict our attention to the weak coupling regime and deformations that involve only the single trace operators of the adjoint fields. To leading order in λ, R_X has the classical value $\frac{1}{2}$. Hence, any superpotential deformation of the form

$$\Delta W = \mathrm{Tr}\, X_{i_1}\cdots X_{i_N}, \qquad N = n + n' \tag{5.27}$$

---

[5]A dual rank of this universal form, independent of whether n is even or odd, is further motivation for the postulated extension of the inequality (5.18) to n even.

with n insertions of X and n' insertions of X' will be relevant as long as

$$\frac{2n'}{3} + \frac{n}{2} < 2 \ \Leftrightarrow \ 4n' + 3n < 12 \ . \tag{5.28}$$

This inequality allows several possibilities.

The linear deformation by $\mathrm{Tr}\,X$ gives F-term equations that cannot be solved, hence it is discarded. The linear deformation by $\mathrm{Tr}\,X'$ is, however, allowed.

There are three quadratic deformations giving rise to the following RG flows

$$\Delta W = \ \mathrm{Tr}\,X'^2 , \quad \widehat{\mathsf{E}} \to \widehat{\mathsf{A}} , \tag{5.29a}$$

$$\Delta W = \ \mathrm{Tr}\,X^2 , \quad \widehat{\mathsf{E}} \to \mathsf{A}_3 , \tag{5.29b}$$

$$\Delta W = \mathrm{Tr}\,XX' , \quad \widehat{\mathsf{E}} \to \mathrm{CS} - \mathrm{SQCD} \ . \tag{5.29c}$$

Finally, the inequality (5.28) allows the cubic deformation

$$\Delta W = \mathrm{Tr}\,X^2 X' , \qquad \widehat{\mathsf{E}} \to \mathsf{D}_4 \ . \tag{5.30}$$

The deformation $\Delta W = \mathrm{Tr}\,X^3$ is equivalent to $\mathrm{Tr}\,X^2 X'$ via a change of variables.

It is natural to expect that as we increase the coupling λ the R-charge R_X will receive more and more negative contributions from the gauge interactions allowing for higher degree relevant superpotential deformations. It is impossible, however, to determine if and when this happens without more detailed information. In this respect, it is worth pointing out that in four dimensions the only (independent) higher degree deformations (becoming relevant at some range of parameters) are [27]

$$\mathsf{E}_6 \ : \ \Delta W = \mathrm{Tr}\,X^4 , \tag{5.31a}$$

$$\mathsf{E}_7 \ : \ \Delta W = \mathrm{Tr}\,X^3 X' , \tag{5.31b}$$

$$\mathsf{E}_8 \ : \ \Delta W = \mathrm{Tr}\,X^5 \ . \tag{5.31c}$$

It is an interesting problem to determine if there is a similar pattern in our CSM theories in three dimensions.

5.5 Comments on mesonic deformations

Our list of RG flows from the $\widehat{\mathsf{O}}$ theory above is certainly not exhaustive and admits more possibilities. Another large class of RG flows is generated by superpotential deformations involving mesonic operators.

An example is provided by the superpotential deformation

$$\Delta W = \widetilde{Q}_i X Q^i \tag{5.32}$$

in the $\widehat{\mathsf{O}}$ theory. Classically this cubic deformation is relevant and leads to a new fixed point which, again following the four-dimensional nomenclature of [27], we will call $\widehat{\mathsf{O}}_{\mathrm{M}}$. Further deformations of this theory by single trace chiral operators are possible and will be discussed in the next section.

Another possibility, which is visible at weak coupling, involves the quartic meson operators $\widetilde{Q}X^2Q$ and $\widetilde{Q}XX'Q$. These operators are classically marginal, but receive negative anomalous dimensions and generate flows that can be described in perturbation theory as in ref. [12].

In general, deformations by higher order mesonic operators, e.g. $\widetilde{Q}X^\ell Q$, may be possible, but precise knowledge of whether and when these operators can become relevant depends on information about the R-charge R_Q for which we have not been able to say much in this paper.

6   RG flows from the $\widehat{\mathsf{O}}_{\mathrm{M}}$ theory

6.1   Input from a Hanany-Witten setup

It is instructive to consider a straightforward generalization of the brane configuration appearing in figure 1 where instead of one NS5-brane and one $(1,k)$ fivebrane bound state we consider n NS5-branes and n' $(1,k)$ fivebrane bound states $(n, n' = 1, 2, \ldots)$. To summarize the configuration we have

$$
\begin{array}{rcl}
n\ NS5 & : & 0\ 1\ 2\ 3\ 4\ 5 \\
n'\ (1,k) & : & 0\ 1\ 2\ \begin{bmatrix} 3 \\ 7 \end{bmatrix}_\theta\ 8\ 9 \\
N_c\ D3 & : & 0\ 1\ 2\ |6| \\
N_f\ D5 & : & 0\ 1\ 2\ 7\ 8\ 9
\end{array}
\tag{6.1}
$$

with the angle θ given by eq. (2.3).

The low energy effective field theory that describes the dynamics of this system is a $\mathrm{U}(N_c)$ $\mathcal{N} = 2$ CSM theory at level k coupled to N_f (anti)fundamentals Q^i, $\widetilde{Q}_i$ and two adjoint chiral superfields X, X'. As in section 4, the fields X, X' are present to describe fluctuations of the D3-branes along the (89) and the (45) planes respectively. There is also a non-trivial superpotential [39]

$$
W_{n,n'} = \frac{g_0}{n+1} \operatorname{Tr} X^{n+1} + \frac{g'_0}{n'+1} \operatorname{Tr} X'^{n'+1} + \sum_{i=1}^{N_f} m \widetilde{Q}_i X' Q^i \ .
\tag{6.2}
$$

The third mesonic superpotential interaction encodes an important difference between the X and X' fields. When we displace the n' $(1,k)$ bound states along the (45) plane, leaving the D5-branes fixed, we make the quark multiplets Q and $\widetilde{Q}$ massive with the same mass of order $\langle X' \rangle$. This effect is accounted for by the Yukawa superpotential coupling m. Its presence breaks the global $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f)$ flavor symmetry to a diagonal $\mathrm{SU}(N_f)$.

The analysis of the vacuum structure of this configuration suggests that the matrices X, X' can be diagonalized independently [39]. Ref. [40] proposed that the superpotential $W_{n,n'}$ includes an additional quartic coupling $\operatorname{Tr}[X, X']^2$ for which we will have little to say here.

From the brane configuration and the s-rule of brane dynamics we read off the following condition for the existence of a supersymmetric vacuum

$$
N_c \leq n N_f + n n' k \ .
\tag{6.3}
$$

As a trivial check, for $n' = 1$ the field X' is massive. In that case, it can be integrated out to obtain the A_{n+1} theory and eq. (6.3) reproduces eq. (4.11).

Another interesting piece of information that we obtain from this brane configuration is Seiberg duality in a large class of 2-adjoint CS-SQCD theories. By moving the N_f D5-branes and the n' $(1, k)$ bound states along the x^6 direction past the n NS5-branes we obtain a configuration similar to the one appearing in figure 1(b). This configuration comprises now of nN_f flavor D3-branes and $n(N_f + n'k) - N_c$ color D3-branes and realizes a dual magnetic description of the original theory. This description is provided by an $\mathcal{N} = 2$ CSM theory with gauge group $\mathrm{U}(n(N_f + n'k) - N_c)$, CS level k and the following matter content: N_f quark pairs $q_i, \tilde{q}^i$, two adjoint chiral superfields Y, Y' and n magnetic meson fields M_ℓ ($\ell = 1, \ldots, n$), each of which is an $N_f \times N_f$ matrix. There is a non-trivial superpotential

$$\widetilde{W}_{n,n'} = \frac{\tilde{g}_0}{n+1} \operatorname{Tr} Y^{n+1} + \frac{\tilde{g}_0'}{n'+1} \operatorname{Tr} Y'^{n'+1} + \sum_{i=1}^{N_f} \widetilde{m} \tilde{q} Y' q + \sum_{\ell=1}^{n} M_\ell \tilde{q} Y^{n-\ell} q \qquad (6.4)$$

with a possible $\operatorname{Tr}[Y, Y']^2$ term as in the electric theory. A four-dimensional analog of this duality was formulated in [39].

The theories appearing in this context can be regarded as deformations of the 2-adjoint $\widehat{\mathsf{O}}_{\mathrm{M}}$ CSM theories defined in the previous section. In what follows we will explore some of the consequences that the above statements have for their dynamics.

6.2 Special case: $n = 1$, $n' \geq 1$

We mentioned that the special case $n \geq 1$, $n' = 1$ reduces (by integrating out the massive X' field) to the A_{n+1} theories which were analyzed before. Another interesting special case is the one with $n = 1$ and $n' > 1$. In this case the superfield X is massive and can be integrated out. Then, one is left with an 1-adjoint CS-SQCD theory with superpotential

$$W_{n'+1} = \frac{g_0'}{n'+1} \operatorname{Tr} X'^{n'+1} + \sum_{i=1}^{N_f} m_i \widetilde{Q}_i X' Q^i . \qquad (6.5)$$

In the brane construction all m_i are equal.

We can view this theory as another deformation of the $\widehat{\mathsf{A}}$ theory. Adding the classically relevant superpotential interaction $\widetilde{Q} X' Q$ to the Lagrangian we flow towards a new set of IR fixed points, which we will call collectively $\widehat{\mathsf{A}}_{\mathrm{M}}$. Then, we deform further to a new set of theories $\mathsf{A}_{\mathrm{M},n'+1}$ by adding the superpotential interactions $\operatorname{Tr} X'^{n'+1}$. Notice that the special case $n' = 1$ with $g_0' = -\frac{k}{4\pi}$ reproduces the theory (3.7) which flows to an $\mathcal{N} = 3$ fixed point with a quartic superpotential for the quarks.

The s-rule derived condition for the existence of a supersymmetric vacuum in the $\mathsf{A}_{\mathrm{M},n'+1}$ theories is

$$N_c \leq N_f + n'k . \qquad (6.6)$$

For $N_c > N_f$, i.e. $x > 1$, it implies that as we increase the coupling λ in the $\widehat{\mathsf{A}}_{\mathrm{M}}$ theory the R-charge of X', $R_{X'}$, decreases while more and more single trace operators $\operatorname{Tr} X'^{n'+1}$

are becoming sequentially relevant. The value of the coupling where $\operatorname{Tr} X'^{n'+1}$ becomes marginal is

$$\lambda^*_{n'+1} < \lambda^{\mathrm{SUSY}}_{n'+1} = \frac{n'x}{x-1} \; . \tag{6.7}$$

A picture similar to the one depicted in figure 4 for the A_{n+1} theory is emerging with an important difference. Assuming $x > 1$, a spontaneous breaking of supersymmetry occurs in the present case for arbitrarily large n'. Therefore, $R_{X',\lim} = 0$ and any operator $\operatorname{Tr} X'^{n'+1}$ can become relevant as long as we make the coupling λ large enough. Furthermore, when all m_i are equal the R-charges of the quarks Q^i, $\widetilde{Q}_i$ are also equal and can be denoted by a single function R_Q. In the $\widehat{\mathsf{A}}_{\mathrm{M}}$ theory the mesonic superpotential interaction is marginal, hence there is a simple relation between R_Q and $R_{X'}$

$$R_Q = 1 - \frac{1}{2}R_{X'} \; . \tag{6.8}$$

Seiberg duality relates this theory to a $\mathrm{U}(N_f + n'k - N_c)$ magnetic version with a single adjoint chiral superfield Y' and N_f pairs of quarks q_i, $\tilde{q}^i$. The magnetic superpotential is

$$\widetilde{W}_{n'+1} = \frac{g'_0}{n'+1} \operatorname{Tr} Y'^{n'+1} + \sum_{i=1}^{N_f} \widetilde{m}_i \tilde{q}^i Y' q_i \; . \tag{6.9}$$

Comparing with the dual superpotential (4.12) we observe that the n' elementary meson superfields M_ℓ are absent. Duality in this case acts in a self-similar way exchanging the rank of the gauge groups $N_c \leftrightarrow N_f + n'k - N_c$ but not the form of the interactions. This generalizes the example presented in section 3.2.1.

We can also understand this duality as an m_i-deformation of Seiberg duality in the $\mathsf{A}_{n'+1}$ case. From this point of view the magnetic theory has gauge group $\mathrm{U}(n'(N_f+k)-N_c)$ and superpotential

$$\widehat{W}_{n'+1} = \frac{g'_0}{n'+1} \operatorname{Tr} Y'^{n'+1} + \sum_{\ell=1}^{n'} M_\ell \tilde{q} Y'^{n'-\ell} q + \sum_{i=1}^{N_f} m_i (M_2)_i^i \; . \tag{6.10}$$

Repeating the analysis of appendix A in [39] we recover the dual description presented above. Many degrees of freedom are massive in the presence of the last term in (6.10) and by integrating them out we recover the dual gauge group $\mathrm{U}(N_f + n'k - N_c)$ and the superpotential (6.9).

6.3 Consequences for R-charges and RG flows

In the general (n, n') case the condition for the existence of a supersymmetric vacuum can be written in terms of the parameters λ, x as

$$\lambda \le \lambda^{\mathrm{SUSY}}_{n,n'} = \frac{nn'x}{x-n} \; . \tag{6.11}$$

There is spontaneous supersymmetry breaking if $x > n$ and $\lambda^{\mathrm{SUSY}}_{n,n'}$ is the maximum value of the coupling.

Once again, this property shows that as we increase λ in the undeformed theory, the R-charges R_X and $R_{X'}$ decrease making more and more single trace operators relevant. At sufficiently large coupling, beyond a critical value $\lambda^*_{n,n'} < \lambda^{\text{SUSY}}_{n,n'}$, the operator $\alpha_n \operatorname{Tr} X^{n+1} + \alpha'_{n'} \operatorname{Tr} X'^{n'+1}$ becomes relevant in the 2-adjoint $\widehat{\mathsf{O}}_M$ theory and drives it to a new set of IR fixed points. From this submanifold of fixed points further deformations with lower power (more relevant) single trace operators is possible. This picture implies a vast set of fixed submanifolds and RG flows connecting them. It would be interesting to obtain a better understanding of these theories.

In conclusion, we find that the $\mathcal{N} = 2$ Landau-Ginzburg-like CSM theories in this section exhibit a rich structure that bears many similarities with the structure familiar from analogous four-dimensional $\mathcal{N} = 1$ SQCD theories. Several features, however, are new in three dimensions compared to the four-dimensional case. For example, there are situations where the R-charges decrease more in three dimensions and the destabilization of the supersymmetric vacuum becomes more efficient. For instance, comparing the models with superpotential (6.5) in three and four dimensions we detect a region of parameters without a supersymmetric vacuum in three dimensions, but no such region in four dimensions [39]. Also, in our three-dimensional 2-adjoint $\widehat{\mathsf{O}}_M$ theories we detect a large set of relevant deformations. A quick calculation of R-charges with a-maximization techniques in the four-dimensional $\widehat{\mathsf{O}}_M$ theory reveals that both R-charges R_X and $R_{X'}$ asymptote to a finite value around $\frac{1}{2}$ at large x allowing for only a limited set of relevant deformations.

7 Discussion

In this paper we considered $\mathcal{N} = 2$ Chern-Simons theories with $U(N_c)$ gauge group coupled to N_f pairs of chiral superfields in the (anti)fundamental representation and zero, one or two chiral superfields in the adjoint. In the absence of superpotential interactions these theories are classically and quantum mechanically superconformal (at least within a range of parameters). Superpotential interactions can be added to generate RG flows towards new IR fixed points. The resulting theories can be viewed as three-dimensional generalizations of $\mathcal{N} = (2, 2)$ Landau-Ginzburg models in two dimensions and bear many similarities with $\mathcal{N} = 1$ (adjoint) SQCD theories in four dimensions.

Using properties like spontaneous breaking of supersymmetry and Seiberg duality we obtained:

(1) a list of semi-quantitative non-perturbative features of $U(1)_R$ symmetries,

(2) a web of RG flows,

(3) the postulation of new interacting fixed points partially admitting an ADE classification,

(4) interesting parallels between three- and four-dimensional gauge theories.

In the process, we argued for a set of new examples of Seiberg duality in three-dimensional CSM theories.

Our discussion provides an ample demonstration of the rich dynamics of Chern-Simons theories coupled to matter. It is of intrinsic interest to develop exact analytic methods that will allow us to study these properties further and beyond perturbation theory.

One can think of several applications in string/M theory. For example, the low-energy dynamics of N M2-branes in flat space is described by a quiver $U(N) \times U(N)$ CSM theory at level 1 with enhanced $\mathcal{N} = 8$ supersymmetry [13]. This theory is strongly coupled. In this and other cases, CSM theories have a dual gravitational description in string or M theory. In all these cases, knowledge about the strong coupling dynamics of the CSM theories is useful not only per se but also for the dual four-dimensional quantum gravity description.

Some related more concrete questions that arise from this work are as follows.

What determines the R-symmetry in $\mathcal{N} = 2$ supersymmetric CSM theories? In three-dimensional superconformal field theories with $\mathcal{N} = 2$ supersymmetry we would like to know if there is an exact analytic method that determines the $U(1)_R$ symmetry. We have seen that this symmetry can receive large non-perturbative contributions. In four-dimensional gauge theories with the same amount of supersymmetry, $i.e.$ $\mathcal{N} = 1$ supersymmetry, the exact $U(1)_R$ symmetry is determined by a-maximization [20]. This method boils down to maximizing a function a that can be expressed as a linear combination of 't Hooft anomalies. The method relies heavily on the ability to identify the candidate R-symmetries in the weakly coupled UV regime, so sometimes more than just weak coupling data is needed to make this method practical. In some cases, this extra information is provided by Seiberg duality [35].

It is natural to ask if there is a similar principle at work in three-dimensional gauge theories with $\mathcal{N} = 2$ supersymmetry, and more specifically in $\mathcal{N} = 2$ CSM theories. Anomalies of continuous symmetries are absent in three dimensions, so if there is an analog of a in three dimensions it will be expressible in a different way.

In four dimensions the function a has been conjectured to be a good candidate for a c-function [20, 41, 42] (see, however, [43]), $i.e.$ a function that is positive and monotonically decreasing along RG flows — Zamolodchikov's c-function in two-dimensional quantum field theory being the prototype example [44]. One wonders whether a tentative a function in three dimensions would also be a good candidate for a c-function. Defining a c-function in three dimensions is a notoriously difficult problem (for work related to this problem see [45, 46]).

Another interesting question is whether we can relate the $\mathcal{N} = 2$ CSM theories to two-dimensional quantum field theories and thus obtain some answers to the above questions from a two-dimensional perspective. The $\mathcal{N} = 2$ Chern-Simons theory with gauge group G becomes, after integrating out the fermions, a bosonic Chern-Simons theory (1.1) at the shifted level

$$k' = k - \frac{h}{2} \operatorname{sgn}(k) \tag{7.1}$$

where k is the $\mathcal{N} = 2$ CS level and h the dual Coxeter number of the gauge group G [30]. This theory, which is a topological quantum field theory, is known to be equivalent to the

(chiral) WZW model with gauge group G and level k' [7].[6] We may ask whether a more general 2d/3d connection persists for CS theories coupled to matter, *e.g.* when the three-dimensional CS theory is placed on a manifold with a two-dimensional boundary. This question is also relevant for the dynamics of M2-branes ending on an M5-brane. In fact, this may be a way to capture the dynamics of the self-dual string on the M5-brane worldvolume from the boundary dynamics of the CSM theory that lives on the M2-brane worldvolume.

RG flows and the ADE classification. In section 5 we presented a subclass of RG flows which appear to admit an ADE classification. It would be interesting to establish the precise range of parameters where these RG flows take place and prove the assertion that the theories that describe the IR dynamics of these flows are superconformal. In some cases, *e.g.* the case of the E_n theories, the precise range of the parameter n needs to be determined.

Assuming that a complete ADE classification takes place in the above subclass of RG flows, it would be interesting to explore if there is a deeper connection with other cases where the ADE classification occurs. For example, in two-dimensional $\mathcal{N} = (2,2)$ theories the ADE superpotentials are special because they lead to the $\hat{c} < 1$ minimal models in which all elements of the chiral ring are relevant operators [48, 49]. Perhaps some of the well known results in two-dimensional $\mathcal{N} = (2,2)$ superconformal field theories, *e.g.* the superconformal Poincare polynomial of R-charges and other properties of critical points [50], can be extended to the three-dimensional $\mathcal{N} = 2$ CSM theories presented in this paper.

Acknowledgments

A Supersymmetries of a Hanany-Witten setup

In this appendix we review the amount of supersymmetry preserved by the configuration (3.8). We will find it convenient to work in M-theory following the analysis of [29]. Without the N_f D5-branes the M-theory version of (3.8) is

$$
\begin{array}{rcl}
N_c\, M2 & : & 016 \\
M5 & : & 012345 \\
(1,k)\, M5' & : & 012 \begin{bmatrix} 3 \\ 7 \end{bmatrix}_\theta 89
\end{array}
\tag{A.1}
$$

---

[6]Incidentally, from this perspective a three-dimensional interpretation of the two-dimensional central charge c was given in [47].

The supersymmetry preserving conditions for these branes are

$$M2 \quad : \quad \Gamma_{016}\epsilon = \epsilon\,, \tag{A.2a}$$

$$M5 \quad : \quad \Gamma_{012345}\epsilon = \epsilon\,, \tag{A.2b}$$

$$M5' \quad : \quad R\Gamma_{012345}R^{-1}\epsilon = \epsilon \tag{A.2c}$$

where R is the rotation matrix ($||$ denotes the 11th direction)

$$R = e^{\frac{\theta}{2}(\Gamma_{2||}+\Gamma_{37})-\frac{\pi}{4}(\Gamma_{48}+\Gamma_{59})} \tag{A.3}$$

and Γ_* are eleven-dimensional Γ-matrices. Ref. [29] shows that this configuration preserves $\mathcal{N} = 2$ supersymmetry in three dimensions.

Now we add N_f M5-branes along

$$\widetilde{M5} \quad : \quad 017 \begin{bmatrix} 4 \\ 8 \end{bmatrix}_\psi \begin{bmatrix} 5 \\ 9 \end{bmatrix}_\psi || \,. \tag{A.4}$$

These branes do not reduce the supersymmetry any further. Indeed, the supersymmetry preserving condition for each of these branes is

$$\hat{R}\Gamma_{01789||}\hat{R}^{-1}\epsilon = \epsilon \tag{A.5}$$

with the obvious ψ-dependent rotation matrix. Since

$$\Gamma_{0123456789||} = 1 \;\Rightarrow\; \Gamma_{01789||} = \Gamma_{016}\Gamma_{012345} \tag{A.6}$$

and

$$\Gamma_{012345}\hat{R}^{-1} = \hat{R}\Gamma_{012345}\,, \qquad\qquad \Gamma_{016}\hat{R} = \hat{R}^{-1}\Gamma_{016} \tag{A.7}$$

we deduce that eq. (A.5) follows from the pre-existing conditions (A.2a), (A.2b). Hence, no more supersymmetries are broken by the rotated D5-branes in the configuration (3.8).

B Evidence for Seiberg duality in the D_{n+2} theories

In this appendix we provide some evidence for the duality proposed in subsection 5.3.2. The electric theory is a $\mathrm{U}(N_c)$ $\mathcal{N} = 2$ CSM theory at level k coupled to N_f pairs of (anti)fundamental chiral superfields $Q^i, \widetilde{Q}_i$ and two adjoint chiral superfields X, X' with superpotential

$$W_{\mathsf{D}_{n+2}} = \frac{g}{n+1}\,\mathrm{Tr}\,X^{n+1} + g'\,\mathrm{Tr}\,XX'^2 \,. \tag{B.1}$$

The proposed magnetic theory is a $\mathrm{U}(3n(N_f+k)-N_c)$ $\mathcal{N} = 2$ CSM theory at level k coupled to N_f pairs of (anti)fundamental chiral superfields $q_i, \tilde{q}^i$, two adjoint chiral superfields Y, Y' and $3nN_f^2$ gauge singlet superfields $(M_{\ell s})^i_j$ ($\ell = 1,\ldots,n,\ s = 1,2,3,\ i,j = 1,\ldots,N_f$) with superpotential

$$\widetilde{W}_{\mathsf{D}_{n+2}} = \frac{\tilde{g}}{n+1}\,\mathrm{Tr}\,Y^{n+1} + \tilde{g}'\,\mathrm{Tr}\,YY'^2 + \sum_{\ell=1}^{n}\sum_{s=1}^{3} M_{\ell s}\tilde{q}Y^{n-\ell}Y'^{3-s}q \,. \tag{B.2}$$

The couplings in front of the meson superpotential interactions are kept implicit.

The mesonic part of the magnetic superpotential respects the global $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f)$ symmetry and is such that the composite magnetic mesons

$$\widetilde{\mathcal{M}}_{\ell s} \equiv \tilde{q} Y^{\ell-1} Y'^{s-1} q \,, \qquad \ell = 1, \ldots, n \,, \qquad s = 1, 2, 3 \tag{B.3}$$

are related to the elementary fields $M_{\ell s}$ by Legendre transform. The Legendre conjugate pairs are $M_{\ell s}$ and $\widetilde{\mathcal{M}}_{n+1-\ell,4-s}$. When the corresponding term in $\widetilde{W}_{\mathsf{D}_{n+2}}$ is relevant we should include $M_{\ell s}$ in the spectrum of independent operators and drop $\widetilde{\mathcal{M}}_{n+1-\ell,4-s}$. Then, by Seiberg duality the elementary fields $M_{\ell s}$ are mapped to the electric composite meson superfields $\widetilde{Q} Y^{\ell-1} Y'^{3-s} Q$.

Giving a complex mass to one of the quarks in the electric theory,

$$W_{\mathsf{D}_{n+2}} \to \frac{g}{n+1} \operatorname{Tr} X^{n+1} + g' \operatorname{Tr} X X'^2 + m \widetilde{Q}_{N_f} Q^{N_f} \tag{B.4}$$

we can integrate out the massive quarks $\widetilde{Q}_{N_f}$, Q^{N_f} and flow to a D_{n+2} theory with $N_f - 1$ quark pairs.

On the magnetic side this deformation corresponds to the superpotential

$$\widetilde{W}_{n+2} \to \frac{\tilde{g}}{n+1} \operatorname{Tr} Y^{n+1} + \tilde{g}' \operatorname{Tr} Y Y'^2 + \sum_{\ell=1}^{n} \sum_{s=1}^{3} M_{\ell s} \tilde{q} Y^{n-\ell} Y'^{3-s} q + m (M_{1,1})_{N_f}^{N_f} \,. \tag{B.5}$$

The F-term equation for the meson $(M_{1,1})_{N_f}^{N_f}$ reveals that the Legendre conjugate composite meson $\widetilde{\mathcal{M}}_{n,3}$ acquires a vacuum expectation value

$$\tilde{q}^{N_f} Y^{n-1} Y'^2 q_{N_f} = -m \,. \tag{B.6}$$

Solving the full set of F-term equations and the D-flatness conditions we obtain non-vanishing expectation values for the quarks $\tilde{q}^{N_f}$, q_{N_f} and the adjoint scalars Y, Y'. Before giving the solution we make a short parenthesis to discuss explicitly the D-flatness conditions.

The relevant terms from the $\mathcal{N} = 2$ CSM Lagrangian are

$$\mathcal{L}_D = \frac{k}{2\pi} D_\beta^\alpha \sigma_\alpha^\beta - \sum_{i=1}^{N_f} \left(q_{i\beta}^\dagger (\sigma^2)_\alpha^\beta q_i^\alpha + \tilde{q}^{\dagger i \beta} (\sigma^2)_\beta^\alpha \tilde{q}_\alpha^i - q_{i\beta}^\dagger D_\alpha^\beta q_i^\alpha + \tilde{q}^{\dagger i \alpha} D_\alpha^\beta \tilde{q}_\beta^i \right)$$
$$- [\sigma, Y]_\alpha^{\dagger \beta} [\sigma, Y]_\beta^\alpha - [\sigma, Y']_\alpha^{\dagger \beta} [\sigma, Y']_\beta^\alpha + Y_\alpha^{\dagger \beta} [D, Y]_\beta^\alpha + Y'^{\dagger \beta}_\alpha [D, Y']_\beta^\alpha \,. \tag{B.7}$$

α, β are gauge indices for the fundamental representation and σ, D are scalars in the $\mathcal{N} = 2$ vector mulitplet. The D_β^α act as Lagrangre multipliers whose equations of motion give

$$\sigma_\alpha^\beta = -\frac{2\pi}{k} \left[\sum_{i=1}^{N_f} \left(q_{i\alpha}^\dagger q_i^\beta - \tilde{q}^{\dagger i \beta} \tilde{q}_\alpha^i \right) + [Y^\dagger, Y]_\alpha^\beta + [Y'^\dagger, Y']_\alpha^\beta \right] \,. \tag{B.8}$$

Inserting this expression back into (B.7) we obtain the D-term potential, which we require to vanish.

As an example, we consider the case with $n = 2$, $N_c = 10$, $N_f = 2$ and $k = 1$. The dual gauge group is U(8). A solution that satisfies all the F-term equations and the D-flatness conditions has (for simplicity we set $\tilde{g} = \tilde{g}' = 1$)

$$\tilde{q}_\alpha^{N_f} = - \left(\frac{m}{2}\right)^{1/5} \delta_{\alpha,1}\,, \qquad q_{N_f}^\alpha = - \left(\frac{m}{2}\right)^{1/5} \delta^{\alpha,6}\,, \qquad \text{(B.9a)}$$

$$Y = - \left(\frac{m}{8}\right)^{1/5} \begin{pmatrix} 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\,, \qquad \text{(B.9b)}$$

$$Y' = - \left(\frac{m}{8}\right)^{1/5} \begin{pmatrix} 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\,. \qquad \text{(B.9c)}$$

The solution has the same form as in the four-dimensional D_{n+2} magnetic theory [38].

In the general case the solution takes the form

$$\tilde{q}_\alpha^{N_f} \sim \delta_{\alpha,1}\,, \qquad q_{N_f}^\alpha \sim \delta^{\alpha,3k}\,,$$
$$Y_{\alpha+1}^\alpha \neq 0\,, \quad \text{but} \quad Y_{k+1}^k = Y_{2k+1}^{2k} = 0\,, \quad \text{and} \quad Y_{2k+1}^k \neq 0\,,$$
$$Y'^\alpha_{\alpha+k} \neq 0 \qquad \text{(B.10)}$$

with all other elements zero. The precise values of the non-vanishing elements can be determined as above by solving the F and D-flatness equations.

The above vacuum expectation values Higgs the gauge group from

$$\text{U}(3n(N_f + k) - N_c) \rightarrow \text{U}(3n(N_f + k - 1) - N_c)\,. \qquad \text{(B.11)}$$

At the same time the q_{N_f} and $\tilde{q}^{N_f}$ quarks are eaten by the gauge group and disappear. The adjoint fields Y, Y' break into smaller $\text{U}(3n(N_f+k-1)-N_c)$ matrices and $6n$ fundamentals. $3n-1$ of these fundamentals are eaten by the Higgs mechanism and $3n+1$ of them become massive. In the IR we recover the theory which is expected to be the magnetic dual to the mass deformed electric theory (B.4).

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